CLASSIFICATION OF AFFINE VORTICES

S. VENUGOPALAN AND C. WOODWARD

Abstract. We prove a Hitchin-Kobayashi correspondence for affine vortices generalizing a result of Jaffe-Taubes [16] for the action of the circle on the affine line. Namely, suppose a compact Lie group K has a Hamiltonian action on a Kähler manifold X which is either compact or a vector space with a linear convex action of K, and so that stable=semistable for the action of the complexified Lie group G. Then, for some sufficiently divisible integer n, there is a bijection between gauge equivalence classes of K-vortices with target X modulo gauge and isomorphism classes of maps from the weighted projective line $\mathbb{P}(1,n)$ to X/G that map the stacky point at infinity $\mathbb{P}(n)$ to the semistable locus in X. The results allow the construction and partial computation of the quantum Kirwan map in Woodward [32], and play a role in the conjectures of Dimofte, Gukov, and Hollands [9] relating vortex counts to knot invariants.

Contents

1. Introduction

In 1980, Jaffe-Taubes [16] provided a classification of finite energy vortices with circle group $K = U(1)$ and target $X = \mathbb{C}$. In this paper we generalize their classification to arbitrary compact Lie groups K and fiber bundles whose fiber X is either a compact Kähler manifold or a finite dimensional complex vector space. That is, we classify *affine vortices* with target X, these are pairs (A, u) where A is the connection on the principal bundle $P = \mathbb{C} \times K$, and $u : \mathbb{C} \to P(X) := (P \times X)/K$ is a holomorphic section, such that (A, u) satisfies the vortex equation:

$$*F_A + \Phi(u) = 0. \tag{1}$$

Here F_A is the curvature of A, $\Phi : X \to \mathfrak{k}^\vee$ is the moment map on X for the K-action, and $*$ is the Hodge star for the standard metric on $\mathbb{C}$.

Jaffe-Taubes [16] show that the gauge-equivalence class of a finite energy vortex is completely determined by the zeros of the section u, so that the moduli space of finite energy vortices is a symmetric product. This is a holomorphic description of the space of vortices with target $\mathbb{C}$ which from a more modern perspective may be viewed as a *Hitchin-Kobayashi correspondence*. Roughly speaking the correspondence takes a gauge-theoretic functional such as the Yang-Mills functional and shows that minima correspond to stable orbits. The first example of such a correspondence by Narasimhan and Seshadri [22] shows that stable holomorphic bundles over a Riemann surface correspond to irreducible unitary representations of the fundamental group. Donaldson [10] reproves the Narasimhan-Seshadri theorem in a differential-geometric setting, replacing irreducible unitary representations by an equivalent object - the minima of the Yang-Mills functional. The extension of this result to higher dimensional base manifold involved replacing a Yang-Mills connection by a Hermitian-Einstein connection. A holomorphic vector bundle admits a Hermitian-Einstein connection if and only if it is stable - this was proved for Kähler surfaces in [11] and for general compact Kähler manifolds by [27].

In the case of vortices one is interested in the study of holomorphic vector bundles over compact Kähler manifolds with additional data, for example a prescribed holomorphic section. Vortices are the zeros of the *vortex functional* given by the norm-square of (1). The zeros of the vortex functional correspond to minima of the Yang-Mills-Higgs functional - also called the energy functional for gauged holomorphic pairs. Bradlow's paper defines a stability condition for such objects and relates it to the existence of zeros of the vortex functional. Results in Bradlow [4] are used to investigate the moduli space of finite energy vortices in [5], [2] and [3]. In case of line bundles, [3] provides a complete description, which is a version of [16] in the case when the base when is a compact Kähler manifold. [21] generalized [4] by allowing the fiber to be a Kähler Hamiltonian manifold. All these Hitchin-Kobayashi results are infinite-dimensional versions of the abstract setting laid out in Kempf-Ness [17] and Kirwan [18], the main idea being that the symplectic quotient coincides with the geometric invariant theory quotient.

Our main result is the following. Recall that if C is a complex curve, then by a definition of Deligne-Mumford the category of morphisms from C to the *quotient stack* X/G is the category of pairs of a principal G-bundle over C together with a section of the associated fiber bundle $P \times_G X$. It contains as a proper open substack the git quotient $X /\!\!/ G$, here defined as the stack-theoretic quotient of the semistable locus by the action of G.

Theorem 1.1. (Classification of affine vortices) *Let X, G, K be as above, and let X/G denote the quotient stack. Let n be a positive integer such that for any $x \in X^{\mathrm{ss}}$, the order of the stabilizer group $|G_x|$ divides n. Then there exists a bijection between gauge equivalence classes of affine K-vortices with target X, and isomorphism classes of morphisms from $\mathbb{P}(1, n)$ to X/G such that $\mathbb{P}(n)$ maps to $X /\!\!/ G$.*

A statement of the result which does not use stack language is given in Theorem 2.10 below. In the case that G is a torus acting on a finite dimensional complex vector space X, bundles on $\mathbb{P}(1,n)$ and sections of the associated vector bundle can be classified explicitly:

Corollary 1.2. (Classification of affine vortices in the toric case) *Suppose that G is a torus acting on a finite dimensional complex vector space X with weights $\mu_1,\ldots,\mu_k$ contained in an open half space, and spanning $\mathfrak{g}^\vee$. Then there is a bijection between affine vortices and isomorphism classes of tuples of polynomial maps $u = (u_1,\ldots,u_k) : \mathbb{C} \to X$ satisfying*

(a) *the degree of u_j is at most $\langle \mu_j, d\rangle$ for each $j = 1,\ldots,k$; and*
(b) *if*

$$u(\infty) = \left(u_j(\infty) := \begin{cases} u_j^{(\langle \mu_j, d\rangle)}/\langle \mu_j, d\rangle! & \langle \mu_j, d\rangle \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases} \right)_{j=1}^{k}$$

denotes the vector of leading order coefficients ($\langle \mu_j, d\rangle$-th derivatives) with integer exponents, then $u(\infty) \in X^{\mathrm{ss}}$.

Two such tuples are isomorphic if they are related by the action of G.

Example 1.3. (a) (Jaffe-Taubes classification) If $X = \mathbb{C}$ with $\mu_1 = 1$, then affine vortices of class d correspond to polynomials of degree exactly d up to the action of scalar multiplication, hence classified by their zeroes. This recovers the Jaffe-Taubes [16] result.

(b) (Matrix-valued vortices and quot schemes) If $X = M_n(\mathbb{C})$, the space of $n \times n$ matrices and $G = GL_n$, the semistable locus consists of invertible matrices and the action on the semistable locus is free. The above theorem gives a classification of vortices according to the following data: By Grothendieck's theorem [15], any vector bundle on $\mathbb{P}^1$ splits as a sum of line bundles

$$P \times_{GL_n(\mathbb{C})} \mathbb{C}^n \cong \mathcal{O}_{\mathbb{P}^1}(\lambda_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(\lambda_n), \quad \lambda_1 \geq \ldots \geq \lambda_n.$$

The associated X bundle is then $P(X) = \mathcal{O}_{\mathbb{P}^1}(\lambda_1)^{\oplus n} \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(\lambda_n)^{\oplus n}$. A section u of $P(X)$ may be viewed as a matrix-valued function on $\mathbb{C}$. The semistability condition at infinity is then the condition that the leading order terms of u form an invertible matrix. Thus u defines a morphism of sheaves which is generically an isomorphism, providing a connection to the quot schemes considered in [1].

A recent paper of Xu [33] complements our work. It proves similar results as this paper for $U(1)$-vortices with fiber $X = \mathbb{C}^m$ using results of [4]. It also shows a correspondence between compactifications of the space of affine vortices modulo gauge on the one side and the space of gauged holomorphic maps over $\mathbb{P}^1$, that are semistable at ∞.

The correspondence described here is partly motivated by a certain quantization of the Kirwan map that arises in the study of Gromov-Witten invariants of geometric invariant theory quotients. Namely Kirwan [18] constructs a map from the

equivariant cohomology $H_G(X, \mathbb{Q})$ to the cohomology of the quotient $H(X/\!\!/G, \mathbb{Q})$. A Gromov-Witten generalization of [18] called the quantum Kirwan map, suggested by Gaio-Salamon [13], maps the equivariant quantum cohomology of X to the quantum cohomology of the quotient $X/\!\!/G$. Affine vortices come up here: the quantum Kirwan is defined by counting affine vortices on X. The work of the second author [23], [32] generalizes the manifolds for which this map is defined - it removes monotonicity and asphericity assumptions on X. Also, the group action can have finite stabilizers on the zero-level set of the moment map, i.e. the symplectic quotient is an orbifold. This paper is part of that project. It provides an algebraic description for the moduli space of vortices, and from there, the quantum Kirwan map can be defined using Behrend-Fantechi machinery. A very important result in this context is a compactification for the space of affine vortices modulo gauge proved by Ziltener ([34], [35]).

An additional recent motivation arises from *knot-invariants via vortex counting* conjectures of Dimofte, Gukov, and Hollands [9], in which the equivariant index (defined via localization) of the moduli space of affine vortices is conjectured to be a certain knot invariant. Our results allow the identification of the moduli space of affine vortices with the *quasimap* spaces discussed in, for example, Bertram, Ciocan-Fontanine, and Kim [1]. The space of matrix-valued vortices in Example 1.3 appear as the relevant space of vortices for a torus knot in the vortex counting conjectures of Dimofte, Gukov, and Hollands [9].

The proof of the main result Theorem 1.1 uses the Hitchin-Kobayashi result for vortices on a compact base manifold with boundary proved by the first author [29], along with an implicit function argument outside a compact set. We may then use the method of exhaustion, solving the vortex equation on a succession of balls of increasing radius, to obtain a solution on the whole plane. Doing so involves a number of complications, such as that the sequence of solutions on the balls converges without bubbling.

We briefly sketch the contents of the paper. Section 2 defines gauged holomorphic maps and extends the definitions to the case when the base manifold Σ is an orbifold $\mathbb{P}(1, n)$. Section 3 introduces stacks and explains how gauged holomorphic maps can be seen as maps between stacks. Section 4 extends the Hitchin-Kobayashi result for compact Riemann surface with boundary in [29] to the case when the target is non-compact. The rest of the sections prove the main theorem.

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2. Background and statement of results

Let G be a complex reductive Lie group, so it is the complexification of a maximal compact subgroup K. Let Σ be a Riemann surface and $P \to \Sigma$ a principal K-bundle. Let (X, ω) be a Kähler manifold on which G acts holomorphically.

Definition 2.1. (a) (Hamiltonian actions) A *moment map* is a K-equivariant map Φ such that $\iota(\xi_X)\omega = \mathrm{d}\langle\Phi,\xi\rangle$, $\forall\xi\in\mathfrak{k}$, where $\xi_X \in \mathrm{Vect}(X)$ given by the infinitesimal action of ξ on X. The action of K is *Hamiltonian* if there exists a moment map $\Phi : X \to \mathfrak{k}^*$. Since K is compact, $\mathfrak{k}$ has an Ad-invariant metric. We fix such a metric and $\mathfrak{k}^*$ and so the moment map becomes a map $\Phi : X \to \mathfrak{k}$. We assume X is equipped with a Hamiltonian action and fix the moment map.

(b) (Geometric invariant theory quotients) If X is a polarized projective G-variety, the geometric invariant theory quotient is $X^{\mathrm{ss}}/\sim$ where X^{ss} is the semi-stable locus and $\sim$ is the orbit closure relation. We assume that the *G-action on X^{ss} has finite stabilizers*. So, $X/\!\!/G$ is the orbit space of X^{ss}. From Kempf-Ness [17] and Kirwan [18], the git quotient $X/\!\!/G$ is homeomorphic to the symplectic quotient $X/\!\!/K := \Phi^{-1}(0)/K$. The stable=semistable guarantees that the K action on $\Phi^{-1}(0)$ has finite stabilizers, and that $X^{\mathrm{ss}} = G\Phi^{-1}(0)$. The results in this work apply to compact Kähler manifolds, in these cases, abusing notation, we define $X^{\mathrm{ss}} = G\Phi^{-1}(0)$.

(c) (Affine space of connections) Let (Σ, j) be a Riemann surface and $P \to \Sigma$ a principal K-bundle over it. In this paper, we mostly don't encounter closed Riemann surfaces, so we may assume that $P = \Sigma \times K$. The *space of connections* is then the affine space

$$\mathcal{A}(P) := \mathrm{d} + \Omega^1(\Sigma, \mathfrak{k}).$$

On any bundle over Σ, where the fiber has a K-action, a connection $A = \mathrm{d}+a$ defines a covariant derivative d_A. For example on the bundle $\Sigma \times X$,

$$\mathrm{d}_A := u \mapsto \mathrm{d}u + a_u \in \Omega^1(\Sigma, u^*TX).$$

At a point $x \in \Sigma$, $a_u(x)$ is the infinitesimal action of $a(x)$ at $u(x)$. The *curvature* of a connection $A = \mathrm{d} + a$ is

$$F_A := \mathrm{d}a + [a \wedge a]/2 \in \Omega^2(\Sigma, \mathfrak{k}).$$

The curvature varies with the connection as

$$F_{A+ta} = F_A + t\mathrm{d}_A a + \frac{t^2}{2}[a \wedge a].$$

(d) (Gauge transformations) A *gauge transformation* is an automorphism of P - it is an equivariant bundle map $P \to P$. The group of gauge transformations on P is denoted $\mathcal{K}(P)$. On the trivial bundle $\Sigma \times K$, $k \in \mathcal{K}(P)$ is a map $k : \Sigma \to K$. It acts on a connection $A = \mathrm{d} + a$ as

$$k(A) = \mathrm{d} + (\mathrm{d}kk^{-1} + \mathrm{Ad}_k\, a).$$

Differentiating, we see that the infinitesimal action of $\xi : \Sigma \to \mathfrak{k}$ on A is $-\mathrm{d}_A\xi$.

(e) (Complex gauge transformations) The *complexified gauge group* $\mathcal{G}(P)$ consists of sections $g : \Sigma \to P \times_K G$. Recall that

$$K \times \mathfrak{k} \to G, \quad (k, s) \mapsto ke^{is} \tag{2}$$

is an isomorphism. So, a complex gauge transformation g can be written as $g = ke^{i\xi}$, where $k \in \mathcal{K}(P)$ and $\xi \in \mathrm{Lie}(\mathcal{K}(P)) = \Gamma(P(\mathfrak{k}))$.

Remark 2.2. (Action of complex gauge transformations on connections) The action of $\mathcal{K}(P)$ on the space of connections extends to an action of $\mathcal{G}(P)$. To see this, we assume for the time being that $K = U(n)$ for some n. Consider the vector bundle $E = P \times_{U(n)} \mathbb{C}^n$ equipped with the Hermitian metric induced by the invariant Hermitian metric on $\mathbb{C}^n$. Let $\mathcal{C}(E)$ denote the space of complex structures on E. Given a connection $A \in \mathcal{A}(P)$, let $\overline{\partial}_A := \pi^{0,1} \circ d_A$. Since the base manifold Σ is two-dimensional, $\overline{\partial}_A^2 = 0$ and so $\overline{\partial}_A$ determines a complex structure on E. Conversely, given a holomorphic structure on E, there is a unique connection compatible with that and the Hermitian metric (see [14]). This shows that there is a bijection between $\mathcal{A}(P)$ and $\mathcal{C}(E)$. Now, suppose $K \subseteq U(n)$ for some n. Infinitesimally, this gives an isomorphism

$$T_A\mathcal{A} = \Omega^1(\Sigma, \mathfrak{k}) \to T_C\mathcal{C} = \Omega^{0,1}(\Sigma, \mathfrak{g}), \quad a \mapsto a^{0,1}.$$

The complex structure on $\mathcal{C}$ pulls back to a complex structure on $\mathcal{A}$ given by $J_{\mathcal{A}}a = a \circ j_\Sigma = *a$. The action of the complexified gauge group on $\mathcal{C}$ pulls back to one on $\mathcal{A}$. For any $\xi \in \mathfrak{k}(P)$, the infinitesimal action of $i\xi$ on A is $- * \mathrm{d}_A\xi$. Now consider a general compact Lie group K. For some n, there is an injection $K \subseteq U(n)$. The K-connections can be seen as a subspace of the $U(n)$-connections. The action of $\mathcal{G}(P)$ preserves K-connections, because for $\xi \in \mathfrak{k}(P)$, the infinitesimal action of $i\xi$ on A is $\mathrm{d}_A\xi \circ j_\Sigma$ which is in $\Omega^1(\Sigma, \mathfrak{k})$.

A connection A determines a holomorphic structure on any associated bundle of P, whose fiber is a complex manifold. One such example is $P(X) = P \times_K X$. Let us call this complex structure $\overline{\partial}_A$.

Definition 2.3. (a) (Gauged holomorphic maps) A *gauged holomorphic map* (A, u) from P to X consists of a connection A and a section u of $P(X)$ that is holomorphic with respect to $\overline{\partial}_A$. The space of gauged holomorphic maps from P to X is called $\mathcal{H}(P, X)$.

(b) (Symplectic vortices) A *symplectic vortex* is a gauged holomorphic map that satisfies

$$*F_A + \Phi(u) = 0.$$

(c) (Energy) $\omega_\Sigma \in \Omega^2(\Sigma)$, determining a metric on Σ. The *energy* of a gauged holomorphic map (A, u) is

$$E(A, u) := \int_\Sigma (|F_A|^2 + |\mathrm{d}_A u|^2 + |\Phi \circ u|^2)\omega_\Sigma.$$

In the literature, the energy map E is also called the Yang-Mills-Higgs functional. For $\Sigma = \mathbb{C}$, finite energy symplectic vortices have good asymptotic properties (see [13] section 11, [35]).

Remark 2.4. (Holomorphicity condition in the case of a trivial bundle) When P is the trivial bundle $\Sigma \times K$, the section u is a map $u : \Sigma \to X$. The formula for the

covariant derivative on $P(X)$ becomes

$$\mathrm{d}_A u = \mathrm{d}u + a_u \in \Omega^1(\Sigma, u^*TX).$$

Here $A = \mathrm{d} + a$ and a_u is the infinitesimal action of a on u - i.e. for any $x \in \Sigma$, $a_u(x) = a(x)_{u(x)}$. Then,

$$\overline{\partial}_A u = \overline{\partial} u + a_u^{0,1}.$$

The complexified gauge group acts on $\mathcal{H}(P, X)$ pairwise:

$$g : (A, u) \mapsto (g(A), gu) = ((g^{-1})^* A, gu).$$

It can be verified that this action preserves holomorphicity (see [29]).

The following observation will be useful.

Remark 2.5. On the bundle $\Sigma \times K$, the stabilizer of the trivial connection $A = d$ is the subgroup of holomorphic gauge transformation $g : \Sigma \to G$, since such gauge transformations commute with $\overline{\partial}_A$.

We first describe principal bundles on $\mathbb{P}(1, n)$ and connections on them. We take a unitary viewpoint in this section, as opposed to a holomorphic one (see remark 2.8). For orbifolds, we follow classical definitions [25]. Gauged holomorphic maps on orbifolds are similar to J-holomorphic curves on orbifolds described in [6]. We will not repeat full definitions here, but just describe what the definitions give in the specific case that the orbifold is $\mathbb{P}(1, n)$.

Definition 2.6. (a) (Weighted projective line) The weighted projective space $\mathbb{P}(1, n)$ is the quotient of $\mathbb{C}^2 - \{0\}$ by the action of $\mathbb{C}^\times$ with weights $1, n$. It is covered by two orbifold charts, $\tilde{U}_1$ and U_2 where $\tilde{U}_1 = U_2 = \mathbb{C}$ and the equivalences are :

$$z \sim e^{2\pi i/n} z \qquad z \in \tilde{U}_1$$

$$z^{-n} \sim w \qquad 0 \neq w \in U_2, 0 \neq z \in \tilde{U}_1$$

We refer to $\tilde{U}_1/\sim$ as U_1.

(b) (Clutching construction) We next describe a principal K-bundle $P \to \mathbb{P}(1, n)$. Let $\tilde{P}|_{\tilde{U}_1}$ denote the lift of $P|_{U_1}$ to $\tilde{U}_1$. P is given by trivializations $\tilde{P}|_{\tilde{U}_1} \simeq \tilde{U}_1 \times K$ and $P|_{U_2} \simeq U_2 \times K$ under the equivalences

$$\begin{aligned} &(z, h) \sim (e^{2\pi i/n} z, \mu(z)h) \qquad (z, h) \in \tilde{U}_1 \times K \\ &(z, h) \sim (w, \tau(z)h) \qquad 0 \neq z \in \tilde{U}_1, w \in U_2, w = \frac{1}{z^n}, h \in K \end{aligned} \tag{3}$$

where $\mu : \tilde{U}_1 \to K$ defines a $\mathbb{Z}_n$-action on $\tilde{U}_1 \times K$ and and $\tau : \tilde{U}_1 \backslash \{0\} \to K$ satisfies $\tau(e^{2\pi i/n} z) = \tau(z)\mu(z)^{-1}$. Note that the fiber over the singular point $0 \in \tilde{U}_1$ may not be K, it could just be $K/\mathbb{Z}_n$

(c) (Clutching construction for connections) Let $\sigma_n : \tilde{U}_1 \to \tilde{U}_1,\ z \mapsto e^{2\pi i/n} z$ denote the diffeomorphism giving the action of the generator of $\mathbb{Z}_n$. A connection on $P \to \mathbb{P}(1, n)$ is given by connections on trivializations $\tilde{U}_1 \times K$ and $U_2 \times K$ that satisfy the equivalences (3):

(i) $A|_{\tilde{U}_1}$ satisfies $\sigma_n^* A = \mu(A)$ (viewing μ as a gauge transformation).

(ii) By the above condition, $\sigma_n^*(\tau(A)) = \tau(A)$ on $\tilde{U}_1 \backslash \{0\}$, so it descends to a connection on $\tilde{U}_1 \backslash \{0\}/\mathbb{Z}_n$. We require that this descended connection is $A|_{U_2 \backslash \{0\}}$.

(d) (Clutching construction for gauge transformations) A gauge transformation k on P consists of $\tilde{k}_1 = k|_{\tilde{U}_1} : \tilde{U}_1 \to K$ and $k_2 = k|_{U_2} : U_2 \to K$ satisfying the equivalences (3) -

(i) $\sigma_n^* k_1 = \mu k_1 \mu^{-1}$ and

(ii) $\tau k_1|_{\tilde{U}_1 \backslash \{0\}} \tau^{-1}$ descends to k_2.

Unless, otherwise mentioned, we think of $\mathbb{P}(1,n)$ as $\mathbb{C}$ with an orbifold singularity at ∞. More precisely, the orbifold point is the quotient $\mathbb{P}(n)$ of $\mathbb{C}^*$ by $\mathbb{C}^*$ acting with weight n. As a groupoid this is equivalent to the quotient of a point by $\mathbb{Z}_n$, that is, $B\mathbb{Z}_n$.

Definition 2.7. (Gauged holomorphic maps from the weighted projective line) Let $P \to \mathbb{P}(1,n)$. A *gauged holomorphic map* (A,u) from $\mathbb{P}(1,n)$ to X consists of gauged holomorphic maps on the bundles $\tilde{U}_1 \times K$ and $U_2 \times K$ that satisfy the equivalence conditions (3)

(a) $(A,u)|_{\tilde{U}_1}$ satisfies $\sigma_n^*(A,u) = \mu(A,u)$ (viewing μ as a gauge transformation).

(b) By the above condition, $\sigma_n^*(\tau(A,u)) = \tau(A,u)$ on $\tilde{U}_1 \backslash \{0\}$, so it descends to a gauged holomorphic map on $\tilde{U}_1 \backslash \{0\}/\mathbb{Z}_n$. We require that this descended map is $(A,u)|_{U_2 \backslash \{0\}}$.

Remark 2.8. (Holomorphic viewpoint) A gauged holomorphic map (A,u) can be described by the following data :

(a) A G-equivariant holomorphic map $u : P_{\mathbb{C}} \to X$, where $P_{\mathbb{C}} = P \times_K G$ and its holomorphic structure is given by A

(b) and a section $\sigma : \Sigma \to P_{\mathbb{C}}/K$.

The complex gauge equivalence class of (A,u) is specified by the map u, that is, does not depend on σ. Holomorphic bundles on $\mathbb{P}(1,n)$ can be described in a similar way to the unitary bundles, the only difference is that the transition functions $\mu : \tilde{U}_1 \to G$ and $\tau : \tilde{U}_1 \backslash \{0\} \to G$ will be holomorphic maps. Another observation is that any holomorphic principal bundle over $\mathbb{C}$ is trivial (see remark 19.6 in [8]). So, a gauged holomorphic map can just be specified by $u : \tilde{U}_1, U_2 \to X$ and the transition functions τ and λ.

Definition 2.9. (Weak gauge transformations and gauged holomorphic maps) Fix $p > 2$. We call a gauged holomorphic map (A,u) on $P \to \mathbb{P}(1,n)$ *weak* if it is smooth on $\mathbb{C}$ and on $B_{\tilde{R}} \subset \tilde{U}_1$, which is a neighbourhood of ∞, $(A,u)|_{B_{\tilde{R}}} \in L^p \times W^{1,p}$. A *(complex) gauge transformation* on P is *weakly extendable* if it is smooth on $\mathbb{C}$ and in $W^{1,p}$ in a neighbourhood of ∞. Denote by $\mathcal{G}(P)_{\mathrm{we}}$ the group of weakly extendable gauge transformations.

By the Sobolev embedding theorem, any weakly extendable (A, u) resp. g is continuous and so $u(\infty)$ resp. $g(\infty)$ is well-defined. The following is the analytic version of the main result of the paper.

Theorem 2.10. *Suppose X is Kähler manifold with Hamiltonian action of a compact Lie group K, which is either compact or a finite dimensional vector space with the linear action of K and a proper moment map. Let G be the complexification of K, and suppose G acts locally freely on X^{ss}. Let n be an integer such that for any $x \in X^{\mathrm{ss}}$, $|G_x|$ divides n. Fix $p > 2$.*

Let (A, u) be a gauged holomorphic map from $\mathbb{C}$ to X that extends to a map over some principal bundle $P \to \mathbb{P}(1, n)$, and suppose $u(\infty) \in X^{\mathrm{ss}}$. There is a weakly extendable complex gauge transformation $g \in \mathcal{G}(P)_{\mathrm{we}}$ such that $g(A, u)$ is a smooth finite energy symplectic vortex, which is unique up to left multiplication by a unitary gauge transformation.

Conversely, given any finite energy symplectic vortex, there is a K-bundle $P \to \mathbb{P}(1, n)$ so that (A, u) extends to a weak gauged holomorphic map on P. There is a weakly extendable complex gauge transformation $g \in \mathcal{G}(P)$ so that $g(A, u)$ is smooth over $\mathbb{P}(1, n)$. The gauged holomorphic map $g(A, u)$ is unique up to complex gauge transformations in $\mathcal{G}(P)$.

3. Gauged holomorphic maps as maps between stacks

In the first three subsections of this section, we provide background and define stacks over the category of manifolds, following [19]. In the last subsection we show how the space of gauged holomorphic map from $\mathbb{P}(1, n)$ can be viewed as functors between stacks. This is used to observe to derive theorem 1.1 from the main theorem 2.10.

Definition 3.1. (a) (CFG's) A *category fibered in groupoids (CFG)* over a category C is a functor $\pi : \mathsf{D} \to \mathsf{C}$ such that

(i) Given an arrow $f : \xi' \to \xi$ in C and an object $\tilde{\xi} \in \mathsf{D}$ with $\pi(\tilde{\xi}) = \xi$ there is an arrow $\tilde{f} : \tilde{\xi}' \to \tilde{\xi}$ in D with $\pi(\tilde{f}) = f$ (we say that $\tilde{\xi}'$ is the *pullback* is the pullback of $\tilde{\xi}$ along f).

(ii) Given a diagram $\begin{tikzcd}[column sep=small, row sep=small] \xi'' \arrow[dr, "f"] & \\ & \xi \\ \xi' \arrow[ur, "h"'] & \end{tikzcd}$ in D and a diagram $\begin{tikzcd}[column sep=small, row sep=small] \pi(\xi'') \arrow[dr, "\pi(f)"] \arrow[dd, "g"'] & \\ & \pi(\xi) \\ \pi(\xi') \arrow[ur, "\pi(h)"'] & \end{tikzcd}$ in C

there is a unique arrow $\tilde{g} : \xi'' \to \xi'$ in D making $\begin{tikzcd}[column sep=small, row sep=small] \xi'' \arrow[dr, "f"] \arrow[dd, "\tilde{g}"'] & \\ & \xi \\ \xi' \arrow[ur, "h"'] & \end{tikzcd}$ commute

and satisfying $\pi(\tilde{g}) = g$.

(b) (Fibers of a CFG) For any $C \in \mathrm{Ob}(\mathsf{C})$, the *fiber over* C is the subcategory $\mathsf{D}(C)$ with objects

$$\mathrm{Ob}(\mathsf{D}(C)) := \{\xi \in \mathrm{Ob}(\mathsf{D}) : \pi(\xi) = C\}$$

and morphisms in $\mathrm{Ob}(\mathsf{D}(C))$ that project to Id_C under π.

Remark 3.2. We summarize some results about CFGs, for details, see [19].

(a) Any two pullbacks of $\tilde{\xi} \in \mathrm{Ob}(\mathsf{D})$ along $f \in \mathrm{Mor}(\mathsf{C})$ are isomorphic.
(b) For any $C \in \mathrm{Ob}(\mathsf{C})$, $\mathsf{D}(C)$ is a groupoid.
(c) The collection of all categories fibered in groupoids over the category C is a 2-category.
The 1-morphisms between CFGs $\pi_{\mathsf{D}} : \mathsf{D} \to \mathsf{C}$ and $\pi_{\mathsf{E}} : \mathsf{E} \to \mathsf{C}$ are maps $F : \mathsf{D} \to \mathsf{E}$ that satisfy $\pi_{\mathsf{E}} \circ F = \pi_{\mathsf{D}}$.
Given two 1-morphism $F, F' : \mathsf{D} \to \mathsf{E}$, a 2-morphisms $\alpha : F \implies F'$ is a natural transformation between them. Since $\mathsf{E}(C)$ is a groupoid, any natural transformation is invertible.

In order to define stacks, we need the concept of a descent category. Let Man denote the category of manifolds with morphisms being smooth maps.

Definition 3.3. (Descent category) Let $\pi : \mathsf{D} \to \mathsf{Man}$ be a category fibered in groupoids, M a manifold and $\{U_i\}$ an open cover of M.

(a) An object in the descent category $\mathsf{D}(\{U_i \to M\})$ is $(\{\xi_i\}, \{\phi_{ij}\})$, where $\xi_i \in \mathrm{Ob}(\mathsf{D}(U_i))$, and $\phi_{ij} \colon \mathrm{pr}_2^* \xi_j \simeq \mathrm{pr}_1^* \xi_i$ is an isomorphism in $\mathsf{D}(U_{ij}) = \mathsf{D}(U_i \times_M U_j)$, such that the following cocycle condition is satisfied: for any triple of indices i, j and k, we have the equality

$$\mathrm{pr}_{13}^* \phi_{ik} = \mathrm{pr}_{12}^* \phi_{ij} \circ \mathrm{pr}_{23}^* \phi_{jk} \colon \mathrm{pr}_3^* \xi_k \to \mathrm{pr}_1^* \xi_i$$

where the pr_{ab} and pr_a are projections to the ath and the bth component, and to the ath component respectively. (For example, $\mathrm{pr}_{13} : U_{ijk} \to U_{ik}$ and $\mathrm{pr}_2 : U_{ijk} \to U_j$.) The isomorphisms ϕ_{ij} are called *transition isomorphisms*.
(b) An arrow $\{\alpha_i\} \in \mathrm{Mor}(\mathsf{D}(\{U_i \to M\}))$ between objects $(\{\xi_i\}, \{\phi_{ij}\})$, $(\{\eta_i\}, \{\psi_{ij}\}) \in \mathrm{Ob}(\mathsf{D}(\{U_i \to M\}))$ is the collection of arrows $\alpha_i \colon \xi_i \to \eta_i$ in $\mathsf{D}(U_i)$, with the property that for each pair of indices i, j, the diagram

$$\begin{tikzcd} \mathrm{pr}_2^* \xi_j \arrow[r, "\mathrm{pr}_2^* \alpha_j"] \arrow[d, "\phi_{ij}"] & \mathrm{pr}_2^* \eta_j \arrow[d, "\psi_{ij}"] \\ \mathrm{pr}_1^* \xi_i \arrow[r, "\mathrm{pr}_1^* \alpha_i"] & \mathrm{pr}_1^* \eta_i \end{tikzcd}$$

commutes.

For a manifold M and cover $\{U_i\}$, there is a functor $\mathsf{D}(M) \to \mathsf{D}(\{U_i \to M\})$ given by pull-back.

Definition 3.4. (Stacks over the category of manifolds) A category fibered in groupoids $\pi : \mathsf{D} \to \mathsf{Man}$ is a *stack* if for any manifold M and any open cover $\{U_i \to M\}$, the pullback functor

$$\mathsf{D}(M) \to \mathsf{D}(\{U_i \to M\}$$

is an equivalence of categories.

Remark 3.5. (a) A map between stacks is a 1-morphism between 2 CFGs.

(b) A manifold $M \in \mathsf{Man}$ corresponds to a stack $\underline{M}$ described in the following way: the objects are $Y \xrightarrow{f} M$, where Y is a manifold and f is a smooth map. Morphisms from $Y_1 \xrightarrow{f_1} M$ to $Y_2 \xrightarrow{f_2} M$ are smooth maps $\phi : Y_1 \to Y_2$ such that $f_2\phi = f_1$. The projection $\pi : \underline{M} \to \mathsf{Man}$ is the forgetful functor that maps $Y \xrightarrow{f} M$ to Y. It can be seen that a functor $\underline{M} \to \underline{N}$ is determined by a map $M \to N$.

Definition 3.6. (Fiber products) Let $\pi_X : X \to \mathsf{C}$, $\pi_Y : Y \to \mathsf{C}$ and $\pi_Z : Z \to \mathsf{C}$ be three categories fibered in groupoids over a category C. The *2-fiber product* $Z \times_X Y$ of the diagram $\begin{array}{ccc} & & Y \\ & & \downarrow f \\ Z & \xrightarrow[g]{} & X \end{array}$ is the category with objects

$$(Z \times_X Y)_0 = \left\{ (y, z, \alpha) \in Y_0 \times Z_0 \times X_1 \mid \pi_Y(y) = \pi_Z(z),\ f(y) \xrightarrow{\alpha} g(z) \right\}$$

and morphisms

$$\mathrm{Hom}_{Z\times_X Y}\big((z_1, y_1, \alpha_1), (z_2, y_2, \alpha_2)\big) =$$

$$\left\{ (z_1 \xrightarrow{v} z_2, y_1 \xrightarrow{u} y_2) \;\middle|\; \pi_Y(u) = \pi_Z(v) \in \mathsf{C}_1,\ \begin{array}{ccc} f(y_1) & \xrightarrow{f(u)} & f(y_2) \\ \alpha_1 \downarrow & \circlearrowleft & \downarrow \alpha_2 \\ g(z_1) & \xrightarrow[g(v)]{} & g(z_2) \end{array} \in X_1 \right\}$$

together with the functor $\pi : Z \times_X Y \to \mathsf{C}$ defined by

$$\pi((z, y, \alpha)) = \pi_Z(z) = \pi_Y(y), \quad \pi(v, u) = \pi_Z(v) = \pi_Y(u)$$

Remark 3.7. The diagram $\begin{array}{ccc} Z \times_X Y & \xrightarrow{\mathrm{pr}_2} & Y \\ \downarrow \mathrm{pr}_1 & & \downarrow g \\ Z & \xrightarrow[f]{} & X \end{array}$ does not commute. But there is a natural isomorphism $g \circ \mathrm{pr}_2 \Rightarrow f \circ \mathrm{pr}_1$ which need not be the identity.

From now on, to simplify notation, we denote $\underline{M}$ just by M. We also drop the distinction between a manifold and the stack induced by it.

Definition 3.8. (a) (Atlases for stacks) Let $\mathsf{D} \to \mathsf{Man}$ be a stack. An *atlas* for D is a manifold X and a map $p : X \to \mathsf{D}$ such that for any manifold

M and map $f : M \to \mathsf{D}$, the fiber product $M \times_{\mathsf{D}} X$ is a manifold and the map $\mathrm{pr}_1 : M \times_{\mathsf{D}} X \to M$ is a surjective submersion. For example, let $\mathcal{U} = \bigsqcup U_i \to M$ be an open cover of a manifold M. Then, the map of stacks $p : \underline{\mathcal{U}} \to \underline{M}$ is an atlas.

(b) (Groupoids associated to atlases) Given an atlas $p : X \to \mathsf{D}$, we can define a Lie groupoid G by $\mathrm{Ob}(G) := X$ and $\mathrm{Mor}(G) := X \times_D X$, which is also a manifold. The source and target maps $s, t : \mathrm{Mor}(G) \to \mathrm{Ob}(G)$ are given by projection $X \times_D X \to X$ to the first and second factor respectively. For any Lie groupoid G, there is a category BG, with objects principal G-bundles (see [19] for a definition. If G is a Lie group, this coincides with the regular notion of principal bundle) and morphisms G-equivariant maps. Define a CFG $\pi : BG \to \mathsf{Man}$, where any principle bundle is sent to its base space and the morphisms are sent to their projection on the base spaces. $BG \to \mathsf{Man}$ is a stack and it is isomorphic to X as a stack.

(c) (Orbifolds) A stack D is *Deligne-Mumford* if it admits an atlas $X \to \mathsf{D}$ so that the corresponding groupoid G is proper and étale, i.e. source and target maps s and t are proper local diffeomorphisms. An *orbifold* is a Deligne-Mumford stack (in the category of manifolds). In our example $\mathbb{P}(1, n)$, $G_0 = \mathbb{C} \bigsqcup \mathbb{C}$ and G_1 corresponds to the equivalences (3).

Definition 3.9. (Quotient stacks) Let X be a manifold on which a Lie group G acts holomorphically. The *quotient stack* X/G is defined as:

(a) The objects of X/G are $\{P \to M, P \xrightarrow{f} X\}$, P is a principal G-bundle over manifold M and f is G-equivariant.

(b) An arrow between $\{P_1 \to M_1, P_1 \xrightarrow{f_1} X\}$ and $\{P_2 \to M_2, P_2 \xrightarrow{f_2} X\}$ is a G-equivariant map $h : P_1 \to P_2$ such that $f_1 = f_2 \circ h$.

(c) The functor $\pi : X/G \to \mathsf{Man}$ maps $\{P \to M, P \xrightarrow{f} X\}$ to M and the morphism h to its projection $M_1 \to M_2$.

The following is standard:

Proposition 3.10. *Suppose that G acts locally freely and properly on X. Then the quotient stack X/G admits the structure of an orbifold.*

We may also work in the category $\mathsf{Man}_{\mathbb{C}}$ whose objects are complex manifolds and whose morphisms holomorphic maps. All the previous definitions are the same, but with holomorphic maps instead of smooth maps. In particular,

(a) a complex orbifold is a Deligne-Mumford stack over $\mathsf{Man}_{\mathbb{C}}$; and

(b) if G is a complex Lie group acting on a complex manifold X then the quotient stack X/G is the category of pairs (P, u) where $P \to M$ is a holomorphic bundle and $u : M \to P(X)$ is a holomorphic section.

From now on, all our stacks are over $\mathsf{Man}_{\mathbb{C}}$. We may now re-interpret our definition of gauged holomorphic map from Definition 2.3 in stack language.

Proposition 3.11. *The set of gauged holomorphic maps from $\mathbb{P}(1,n)$ to X modulo complex gauge transformations is in bijection with isomorphism classes of objects of* $\mathrm{Hom}(\mathbb{P}(1,n), X/G)$.

We will prove this after recalling the following version of the Yoneda lemma:

Lemma 3.12 (2-Yoneda : [19] Lemma 4.19)**.** *Let $\mathsf{D} \to \mathsf{C}$ be a category fibered in groupoids (CFG). For any object $X \in \mathsf{C}$, there is an equivalence of categories*

$$\Theta : \mathrm{Hom}_{CFG}(\underline{X}, \mathsf{D}) \to \mathsf{D}(X). \tag{4}$$

$\mathrm{Hom}_{CFG}(\underline{X}, D)$ *denotes the category of maps of CFGs and natural transformations between them.*

Proof of proposition 3.11. Lemma 3.12 proves proposition 3.11 in the case $n = 1$: $\mathrm{Hom}(\underline{\mathbb{P}}^1, X/G)$ is equivalent to $X/G(\mathbb{P}^1)$. The objects in the fiber $X/G(\mathbb{P}^1)$ are $\{P \to \mathbb{P}^1, P \xrightarrow{f} X\}$ where P is a holomorphic G-bundle and f is G-equivariant. The morphisms between $\{P_1 \to \mathbb{P}^1, P_1 \xrightarrow{f_1} X\}$ and $\{P_2 \to \mathbb{P}^1, P_2 \xrightarrow{f_2} X\}$ are isomorphisms and they consist of G-equivariant holomorphic maps between principal bundles $P_1 \xrightarrow{h} P^2$ that satisfy $u_2 \circ h = u_1$ and are identity on the base $\mathbb{P}^1$. These morphisms are precisely complex gauge transformations that take (P_1, u_1) to (P_2, u_2).

For the general case, consider $u \in \mathrm{Hom}(\mathbb{P}(1,n), X/G)$. The pair $\tilde{U}_1, U_2$ described in 2.6 is an atlas for $\mathbb{P}(1,n)$ making it an étale Lie groupoid H, and hence a Deligne-Mumford stack. In the space of morphisms $\mathrm{Mor}(H)$ in the groupoid H, one of the components is $H_1^1 = \{(x, \sigma_n x) : x \in \tilde{U}_1\}$. By pulling back, we have $u : \tilde{U}_1 \to X/G$.

$$\begin{tikzcd} H_1^1 \subset \tilde{U}_1 \times_{\mathbb{P}(1,n)} \tilde{U}_1 \arrow[r, "pr_1"] \arrow[d, "pr_2"] & \tilde{U}_1 \arrow[d, "f"] \\ \tilde{U}_1 \arrow[r, "g"] & \mathbb{P}(1,n) \end{tikzcd}$$

The pull-backs $(g \circ \mathrm{pr}_2)^*\xi$ and $(f \circ \mathrm{pr}_1)^*\xi$ are related by a natural isomorphism, this means there is a $\mu : \tilde{U}_1 \to G$ that satisfies

$$u(\sigma_n x) = \mu(x) u(x) \tag{5}$$

for all $x \in \tilde{U}_1$. Since u is holomorphic, μ is also holomorphic.

Another component of $\mathrm{Mor}(H)$ is $H_1^2 = \{(z, w) \in \tilde{U}_1 \times U_2 : w = \frac{1}{z^n}\}$. In a similar way that leads to a transition map $\tau : \tilde{U}_1 \backslash \{0\} \to G$ that satisfies

$$\tau u(z) = u(w), \tag{6}$$

where $z \in \tilde{U}_1$, $w \in U_2$ and $w = \frac{1}{z^n}$. Since $u|_{\tilde{U}_1}$ and $u|_{U_2}$ are holomorphic, τ is also holomorphic. Equations (5) and (6) imply that $\tau \circ \sigma_n = \tau \mu^{-1}$. This shows that $u \in \mathrm{Hom}(\mathbb{P}(1,n), X/G)$ corresponds to a gauged holomorphic map from $\mathbb{P}(1,n)$ to X. It's easy to see the reverse is true: given a gauged holomorphic map from $\mathbb{P}(1,n)$ to X, we get an element in $\mathrm{Hom}(\mathbb{P}(1,n), X/G)$. Complex gauge transformations correspond to natural isomorphisms between functors in $\mathrm{Hom}(\mathbb{P}(1,n), X, G)$. □

Proposition 3.11 shows that theorem 2.10 yields theorem 1.1.

4. Heat flow for vector space target

In this section we extend the Hitchin-Kobayashi correspondence of [29] to the case of non-compact target. We recall the main result of [29]. If s, p are positive integers such that $s - \frac{1}{p} \geq 0$, we denote by $W^{s,p}_\delta(\Omega) = \{f \in W^{s,p}(\Omega) : f|_{\partial\Omega} = 0\}$ the Sobolev space vanishing on the boundary.

Theorem 4.1. *([29] theorem 4.3.3) Let Σ be a compact Riemann surface with non-empty boundary and (A, u) be a gauged holomorphic curve mapping to a compact Kahler Hamiltonian manifold X so that $*F_A + \Phi(u) = 0$ on $\partial\Sigma$. Then, there is a complex gauge transformation $g = e^{i\xi}$, $\xi \in W^{2,p}_\delta$ (for any $p > 2$) such that $g(A, u)$ is a vortex. Up to unitary gauge equivalence, $g(A, u)$ is the unique vortex in the complex gauge orbit of (A, u).*

In this section, we extend this result to the case when $X = \mathbb{C}^n$, the group action is linear and the moment map is proper. Suppose K acts via a group homomorphism $\rho : K \to U(n)$. A moment map for the action is

$$\Phi(x) := (\mathrm{d}\rho)^* \left(-\frac{\iota}{2} x x^*\right) + \tau$$

Here $\mathrm{d}\rho : \mathfrak{k} \to \mathfrak{u}(n)$ is the differential of ρ at Id, and $(\mathrm{d}\rho)^*$ is its dual with respect to the chosen Ad-invariant inner product on $\mathfrak{k}$ and the inner product $\langle A, B\rangle = \mathrm{tr}(A^*B)$ on $\mathfrak{u}(n)$. The element $\tau \in \mathfrak{k}$ is a central element chosen such that K acts locally freely on $\Phi^{-1}(0)$.

In [29], theorem 4.1 is proved by showing the long-term existence of gradient flow of (A, u) under the functional $(A, u) \mapsto \|*F_A + \Phi(u)\|_{L^2(\Sigma)}$. As $t \to \infty$, the gradient flow converges modulo gauge to a symplectic vortex. All the results assume that the target X is compact.

Lemma 4.2. *Suppose Σ is a compact Riemann surface (possibly with boundary), $X = \mathbb{C}^n$ with G acting linearly. We assume the moment map Φ is proper. Let (A, u) be a gauged holomorphic curve on Σ with the energy bound $E(A, u) \leq k$. Then, there is a compact set $S \subseteq \mathbb{C}^n$, determined only by k, that contains $u(\Sigma)$.*

Proof of lemma 4.2. Define an operator L_x for every $x \in X$,

$$L_x : \mathfrak{k} \to T_X X, \quad \xi \mapsto \xi_X(x) \tag{7}$$

So, given $u : \Sigma \to X$, we obtain a section

$$L_u \in \Gamma(\Sigma, \mathrm{Hom}(\mathfrak{k}, T_u X))$$

of the vector bundle $\mathrm{Hom}(\mathfrak{k}, T_u X)$ on Σ.

Step 1: *Proof of lemma assuming that $\partial\Sigma = \phi$.*

By the energy bound, $\|F(A)\|_{L^2}, \|\Phi(u)\|_{L^2} < k$. Using Uhlenbeck's local theorem ([28]), we can find a cover of Σ, $\cup_\alpha \mathcal{U}_\alpha$ and local trivializations under which the connection A is $\mathrm{d} + a_\alpha$ on $\mathcal{U}_\alpha$ and $\|a_\alpha\|_{H^1(\mathcal{U}_\alpha)} < c_k$. Here c_k is a constant depending only

on k. Suppose, under this trivialization u is given by $u_\alpha : \mathcal{U}_\alpha \to \mathbb{C}^n$. Holomorphicity yields $\overline{\partial} u_\alpha = (a_\alpha)^{0,1}_{u_\alpha}$.

The lemma is proved by using elliptic regularity to obtain a C^0 bound on each u_α. First we produce an L^2 bound on each u_α. Since Φ is a quadratic function on X and is proper, we have

$$\|u\|_{L^2} < c(1 + \|\Phi(u)\|_{L^2}) < c_k.$$

Note that $(a_\alpha)_{u_\alpha}$ is the product of two sections $L_{u_\alpha} \in \Gamma(\Sigma, \mathrm{End}(\mathfrak{k}, T_{u_\alpha} X))$ and $a_\alpha \in \Gamma(\Sigma, \mathfrak{k})$. $|L_x|$ grows linearly with x, so $|L_x| \approx c|\Phi(x)|^{1/2}$. Since $\|\Phi(u_\alpha)\|_{L^2} < k$, $\|L_{u_\alpha}\|_{L^4} < c_k$. By the multiplication theorem (proposition A.5),

$$\|(a_\alpha)_{u_\alpha}\|_{L^{2+\epsilon}(\mathcal{U}_\alpha)} < c_k.$$

Let $\mathcal{U}''_\alpha \Subset \mathcal{U}'_\alpha \Subset \mathcal{U}_\alpha$ be such that $\mathcal{U}''_\alpha$ still cover Σ. We apply interior elliptic regularity twice to obtain

$$\|u_\alpha\|_{W^{1,2}(\mathcal{U}'_\alpha)} \leq c(\|\overline{\partial} u_\alpha\|_{L^2(\mathcal{U}_\alpha)} + \|u_\alpha\|_{L^2(\mathcal{U}_\alpha)}) \leq c_k.$$

By Sobolev embedding, $W^{1,2} \hookrightarrow L^{2+\epsilon}$ and so, the $L^{2+\epsilon}$ norms of u_α are bounded by c_k. Next,

$$\|u_\alpha\|_{W^{1,2+\epsilon}(\mathcal{U}''_\alpha)} \leq c(\|\overline{\partial} u_\alpha\|_{L^{2+\epsilon}(\mathcal{U}'_\alpha)} + \|u_\alpha\|_{L^{2+\epsilon}(\mathcal{U}'_\alpha)}) \leq c_k.$$

By the inclusion $W^{1,2+\epsilon} \hookrightarrow C^0$, $u_\alpha(\mathcal{U}_\alpha)$ is contained in a compact set $S_\alpha \subseteq \mathbb{C}^n$. We can replace S_α by KS_α, which is also compact. In this way we see the compact set is independent of the trivialization we choose over $\mathcal{U}_\alpha$.

Step 2: *When $\partial\Sigma \neq \phi$.*

We use a technique from the proof of lemma 4.2.6 in [29] to construct a gauged holomorphic map on a closed Riemann surface $\tilde{\Sigma}$ that is made up of two copies of Σ. For ease of notation, we assume throughout the proof that $\partial\Sigma$ has one component, the proof works identically for multiple components.

Assume that $B_\epsilon(\partial\Sigma)$ is the ϵ-neighbourhood of $\partial\Sigma$, parametrized by $\{z \in \mathbb{C} : 1 \leq |z| < 1 + \epsilon\}$. There is a trivialization

$$\tau : P|_{B_\epsilon(\partial\sigma)} \to \{z \in \mathbb{C} : 1 \leq |z| < 1 + \epsilon\} \times K,$$

under which A is in radial gauge, i.e. $(\tau_1^{-1})^* A = \mathrm{d} + a_\theta \mathrm{d}\theta$. Consider the bundle

$$\tilde{P} := \left(P \bigsqcup P\right) / \{(x,x) : x \in P_{\partial\Sigma}\}$$

over the Riemann surface $\tilde{\Sigma} := \Sigma \bigsqcup \Sigma / \{(x,x) : x \in \partial\Sigma\}$. At the boundary, the trivialization τ of $P|_{B_\epsilon(\partial\Sigma)}$ extends to a trivialization of $\tilde{P}$:

$$\tilde{P}|_{B_\epsilon(\partial\Sigma)} \simeq \left\{ z \in \mathbb{C} \;\middle|\; \frac{1}{1+\epsilon} < |z| < 1 + \epsilon \right\} \times K.$$

This defines the manifold structure of $\tilde{P}$ close to $\partial\Sigma$. Next, the connection $\tilde{A}$ on $\tilde{P}$ be given by the A on both copies of P. The trace of these connections on $P|_{\partial\Sigma}$ agree, so this is a H^1 connection on $\tilde{P}$. $\tilde{u} : \tilde{\Sigma} \to X$ is defined to be same as u on both copies. Finally, we note that Step 1 applies if (A, u) is in the space $H^1 \times (H^1_{\mathrm{loc}} \cap C^0_{\mathrm{loc}})$,

and $E(\tilde{A}, \tilde{u}) = 2E(A, u) \le 2k$. Step 1 implies $\tilde{u}(\tilde{\Sigma})$ is contained in a compact set of X, and so the same holds for u. $\square$

Lemma 4.2.6 in [29] tells us that if (A_t, u_t) is the gradient flow line of the functional $\|{*F_A} + \Phi(u)\|^2_{L^2(\Sigma)}$ for $t \in [0, T]$, then $E(A_t, u_t)$ decreases with t. Along with the above lemma this implies that if the gradient flow exists till time t, $u_t(\Sigma) \subseteq S$. So, we may assume that the target manifold is S. Therefore the flow exists for all time. We have proved:

Proposition 4.3. *Theorem 4.1 holds in the case when $X = \mathbb{C}^n$, the action of K is linear and the moment map Φ is proper.*

5. From a holomorphic map to a vortex: outside a compact set

In this section we find a complex gauge transformation so that the vortex equation is satisfied outside of a ball. That is,

Proposition 5.1. *Suppose the action of G on X^{ss} has finite stabilizers. Let (A, u) be a gauged holomorphic map from $\mathbb{C}$ to X that extends to one over a principal bundle $P \to \mathbb{P}(1, n)$, and suppose $u(\infty) \in X^{\mathrm{ss}}$. Then there exists a smooth function $s : \mathbb{C} \to \mathfrak{k}$ such that if $e^{is} : \mathbb{C} \to G$ denotes the gauge transformation obtained by exponentiating is then $e^{is}(A, u)$ is a finite energy vortex on $\mathbb{C} \backslash B_R$ for some R. The gauge transformation e^{is} extends continuously to a complex gauge transformation on $P \to \mathbb{P}(1, n)$.*

Definition 5.2. We define the following natural covering spaces of $\mathbb{C} \backslash B_R$. For any $R > 0$, let
$$\tilde{\Sigma}_R := \mathbb{C} \backslash B_{R^{1/n}}$$
be the n-fold cover of Σ_R, with covering map $z \mapsto z^n$. Let
$$\overline{\Sigma}_R = \{re^{i\theta} : r \ge R,\ \theta \in \mathbb{R}/2n\pi\mathbb{Z}\}$$
be the isometric n-fold cover of Σ_R with covering map is $re^{i\theta} \mapsto re^{i(\theta \mod 2\pi)}$.

The metrics differ on $\tilde{\Sigma}_R$ and $\overline{\Sigma}_R$ as follows. Let $\tau : \tilde{\Sigma}_R \to \overline{\Sigma}_R$ be the obvious map that commutes with the covering maps. Let $\omega_{\tilde{\Sigma}_R}$ and $\omega_{\overline{\Sigma}_R}$ be the Euclidean Kähler form on these spaces. Then, $\tau^*\omega_{\overline{\Sigma}_R} = n^2 r^{2n-2}\omega_{\tilde{\Sigma}_R}$. The L^2 norms transform as follows

$$\begin{aligned}
&\alpha \in \Omega^0(\overline{\Sigma}_R) && \int_{\overline{\Sigma}_R} |\alpha|^2 \omega_{\overline{\Sigma}_R} = \int_{\tilde{\Sigma}_R} |\tau^*\alpha|^2 n^2 r^{2n-2} \omega_{\tilde{\Sigma}_R} \\
&\alpha \in \Omega^1(\overline{\Sigma}_R) && \int_{\overline{\Sigma}_R} |\alpha|^2 \omega_{\overline{\Sigma}_R} = \int_{\tilde{\Sigma}_R} |\tau^*\alpha|^2 \omega_{\tilde{\Sigma}_R} \\
&\alpha \in \Omega^2(\overline{\Sigma}_R) && \int_{\overline{\Sigma}_R} |\alpha|^2 \omega_{\overline{\Sigma}_R} = \int_{\tilde{\Sigma}_R} |\tau^*\alpha|^2 n^{-2} r^{-2n+2} \omega_{\tilde{\Sigma}_R}
\end{aligned} \tag{8}$$

At the center of the proof of proposition 5.1 is an implicit function theorem argument. This is part of the following proposition.

Proposition 5.3. *Let $R_0 > 0$ and (A,u) be a gauged holomorphic map over $\overline{\Sigma}_{R_0}$. We assume the following:*

(a) $\|{*F_A} + \Phi(u)\|_{L^2} < \infty$;
(b) *u extends continuously over ∞, i.e there is a point $u(\infty) \in \Phi^{-1}(0)$ so that $\lim_{r\to\infty} u(re^{i\theta}) = u_\infty$ for all $\theta \in [0, 2n\pi)$; and*
(c) *on the trivial bundle $\overline{\Sigma}_{R_0} \times K$, $A = \mathrm{d} + a$, $a \in \Omega^1(\overline{\Sigma}_{R_0}, \mathfrak{k})$ satisfies $\|a\|_{H^1} < \infty$.*

Then, there exists $R \geq R_0$ for which there is a complex gauge transformation $g \in H^2(\overline{\Sigma}_R, G)$ so that $g(A,u)$ is a finite energy vortex on $\overline{\Sigma}_R$. In addition, suppose (A,u) satisfies the symmetry relation $(A,u)(e^{2\pi i}\cdot) = \gamma(A,u)$, where $\gamma \in K$ and $\gamma^n = \mathrm{Id}$, then, g satisfies

$$g(e^{2\pi i}\cdot) = \gamma g(\cdot)\gamma^{-1}. \tag{9}$$

We remark that it is enough to look for a complex gauge transformation of the form $g = e^{i\xi}$. This is because any complex gauge transformation g can be written as $g = ke^{i\xi}$, where $k \in \mathcal{K}$ and $\xi \in \mathrm{Lie}(\mathcal{K})$, and if $g(A,u)$ is a symplectic vortex, so is $e^{i\xi}(A,u)$.

Proof of proposition 5.3. Define the functional

$$\mathcal{F}^R : \Gamma(\mathbb{C}\backslash B_R, \mathfrak{g}) \to \Gamma(\mathbb{C}\backslash B_R, \mathfrak{g}), \quad \xi \mapsto *F_{(\exp i\xi)A,(\exp i\xi)u}.$$

The linearization at ξ

$$D\mathcal{F}^R_\xi(\xi_1) = d^*_{(\exp i\xi)D} d_{(\exp i\xi)D}\xi_1 + u^*\mathrm{d}\Phi(J(\xi_1)_u)$$

extends to a map between Sobolev completions:

$$D\mathcal{F}^R_\xi : H^2(\mathbb{C}\backslash B_R, \mathfrak{g}) \to L^2(\mathbb{C}\backslash B_R, \mathfrak{g}).$$

The Sobolev norm for the spaces $H^s_R := H^s(\mathbb{C}\backslash B_R, \mathfrak{g})$ is defined in the usual way:

$$\|\sigma\|_s := \left(\sum_{i=0}^s \|\nabla^i\sigma\|^2_{L^2(\mathbb{C}\backslash B_R)}\right)^{1/2}$$

where ∇ is the trivial connection.

The proof of the theorem proceeds by showing that

$$D\mathcal{F}^R_0 : \xi_1 \mapsto \mathrm{d}^*\mathrm{d}\xi_1 + u^* d\Phi(J(\xi_1)_u)$$

is onto, since it is 'close' to $\mathrm{d}^*\mathrm{d} + \mathrm{Id}$. Then we'll apply the implicit function theorem to find a solution for $\mathcal{F}^R = 0$ close to $\xi = 0$. For notational convenience, we define an operator $\mathcal{L}_x$ for every $x \in X$,

$$\mathcal{L}_x : \mathfrak{g} \to \mathfrak{g}, \quad \xi \mapsto d\Phi_x(J\xi_x)$$

If K acts locally freely at x, then $\mathcal{L}_x$ is a positive operator.

For proving the symmetry result (9) we define the operator

$$\tau_\gamma : \Gamma(\overline{\Sigma}, \mathfrak{k}) \to \Gamma(\overline{\Sigma}, \mathfrak{k}), \quad \xi \mapsto \mathrm{Ad}_{\gamma^{-1}}\, \xi(e^{2\pi i}\cdot).$$

STEP 1: *There exists R_1 such that for $R \geq R_1$, $D\mathcal{F}^R(0) : H^2_R \to L^2_R$ is onto and has a right inverse Q^R, with $\|Q^R\| < C$ and C is independent of R. If*

$$(A,u)(e^{2\pi i}\cdot) = \gamma(A,u),$$

then Q_R can be chosen so that $\operatorname{Im} Q_R$ is invariant under τ_γ.

Write

$$D\mathcal{F}_0 = \mathrm{d}_A^*\mathrm{d}_A + \mathcal{L}_u = (\mathrm{d}^*\mathrm{d} + \mathcal{L}_{u(\infty)}) + (\mathrm{d}_A^*\mathrm{d}_A - \mathrm{d}^*\mathrm{d}) + (\mathcal{L}_u - \mathcal{L}_{u(\infty)}) \tag{10}$$

We first show that $(\mathrm{d}^*\mathrm{d} + \mathcal{L}_{u(\infty)})$ is onto with a right inverse bounded independently of R. The contributions of the other terms can be controlled by choosing R large enough. The operator $\mathcal{L}_{u(\infty)} : \mathfrak{k} \to \mathfrak{k}$ is positive and so is diagonalizable with positive eigen-values $\lambda_1, \dots, \lambda_m$. Then, the bundle $(\mathbb{C}\backslash B_R) \times \mathfrak{k}$ splits into a direct sum of eigen sections L_i of the bundle. On L_i the operator $\mathrm{d}^*\mathrm{d} + \mathcal{L}_u|_{L_i} = \mathrm{d}^*\mathrm{d} + \lambda_i$. We now choose a right inverse q^R for

$$\mathrm{d}^*\mathrm{d} + \mathrm{Id} : H^2(\mathbb{C}\backslash B_R, \mathbb{C}) \to L^2(\mathbb{C}\backslash B_R, \mathbb{C}).$$

Given $\sigma \in L^2_R$, extend it by 0 in B_R to get $\tilde{\sigma} \in L^2(\mathbb{C}\backslash B_1, \mathbb{C})$. By proposition A.1,

$$(\mathrm{Id} + \mathrm{d}^*\mathrm{d}) : H^2_\delta(\mathbb{C}\backslash B_1, \mathbb{C}) \to L^2(\mathbb{C}\backslash B_1, \mathbb{C})$$

is invertible where

$$H^2_\delta(\Omega) := \{f \in H^2(\Omega) : f|_{\partial\Omega} = 0\}.$$

Let r_R denote restriction of a function from $\mathbb{C}\backslash B_1$ to $\mathbb{C}\backslash B_R$. Define as

$$q_R(\sigma) = r_R(\mathrm{Id} + \mathrm{d}^*\mathrm{d})^{-1}\tilde{\sigma}.$$

Then $\|q_R\| < k$ and k is independent of R. By scaling, we can obtain a right inverse for $\mathrm{d}^*\mathrm{d} + \lambda_i$ also. So, $\mathrm{d}^*\mathrm{d} + \mathcal{L}_{u(\infty)} : W^{2,p}_R \to L^p_R$ has a right inverse $\tilde{Q}_R$ with norm bounded by k/λ, where $\lambda := \min\{1, \lambda_1, \dots, \lambda_m\}$.

Next, we look at the perturbative terms. We can write

$$(\mathrm{d}_A^*\mathrm{d}_A - \mathrm{d}^*\mathrm{d})\xi = *[a \wedge *d\xi] + [d^*a \wedge \xi] + *[a \wedge *[a \wedge \xi]].$$

By the Sobolev multiplication theorem (proposition A.5),

$$\begin{aligned}\|*[a \wedge *\mathrm{d}\xi] + [\mathrm{d}^*a \wedge \xi] + *[a \wedge *[a \wedge \xi]]\|_{L^2_R} &\leq c(\|a\|_{H^1_R} + \|a\|^2_{H^1_R})\|\xi\|_{H^2_R} \\ &\leq \frac{\lambda}{4k}\|\xi\|_{H^2_R}\end{aligned}$$

for small $\|a\|_{H^1_R}$, which can be achieved by taking R large. Consider $\mathcal{L}_u - \mathcal{L}_{u(\infty)}$. As $R \to \infty$, $u(Re^{i\theta}) \to u(\infty)$ uniformly and so $\|\mathcal{L}_u - \mathcal{L}_\infty\|_{C^0(\mathbb{C}\backslash B_R)} \to 0$. For large R,

$$\|\mathcal{L}_u - \mathcal{L}_\infty\|_{C^0(\mathbb{C}\backslash B_R)} < \lambda/4k.$$

And hence, the operator $\|\mathcal{L}_u - \mathcal{L}_\infty\| : H^2_R \to L^2_R$ has norm $\lambda/4k$. So, for a suitable R_1,

$$\|(\mathrm{d}_A^*\mathrm{d}_A - \mathrm{d}^*\mathrm{d}) + (\mathcal{L}_u - \mathcal{L}_{u(\infty)})\| \leq \lambda/2k \leq \frac{1}{2\|\tilde{Q}_R\|}$$

for all $R \geq R_1$. So, for $R \geq R_1$, $d\mathcal{F}_R(0) : \operatorname{Im} \tilde{Q}_R \to L^2_R$ is an isomorphism with the norm of the inverse bounded by $2\|\tilde{Q}_R\| \leq 2k/\lambda$.

To prove symmetry, assume $(A, u)(e^{2\pi i} \cdot) = \gamma(A, u)$. Since $\operatorname{Im} Q_R = \operatorname{Im} \tilde{Q}_R$, it is enough to make $\operatorname{Im} \tilde{Q}_R$ invariant under τ_γ. If $\tilde{Q}_R$ is a right inverse of $\mathrm{d}^*\mathrm{d} + L_{u(\infty)}$, since $u(\infty)$ is fixed by γ, $\tau_\gamma \circ \tilde{Q}_R$ is also a right inverse. And, since $\mathrm{d}^*\mathrm{d} + L_{u(\infty)}$ is a linear operator, we can replace $\tilde{Q}_R$ by its average under τ_γ, given by $\frac{1}{n}\sum_{i=0}^{n-1} \tau_\gamma^i \circ \tilde{Q}_R$. The bound on the norm changes by a constant.

Step 2: *There is a constant $\delta > 0$ such that for $\|\xi\|_{2,p} < \delta$, $\|D\mathcal{F}^R_\xi - D\mathcal{F}^R_0\| \leq \frac{1}{2C}$ (C is the constant in Step 1)*

Write

$$D\mathcal{F}^R_\xi - D\mathcal{F}^R_0 = (d^*_{(\exp i\xi)A} d_{(\exp i\xi)A} - \mathrm{d}^*_A \mathrm{d}_A) + (\mathcal{L}_{(\exp i\xi)u} - \mathcal{L}_u)$$

Consider the first term. Let $(\exp i\xi)A = A + a$. Then,

$$(d^*_{(\exp i\xi)A} d_{(\exp i\xi)A} - \mathrm{d}^*_A \mathrm{d}_A)\xi_1 = *[a \wedge *d\xi_1] + d^*[a \wedge \xi_1] + *[a \wedge *[a \wedge \xi_1]].$$

The multiplication theorem on Euclidean domains (proposition A.5) is bounded by constants independent of the domain (it depends only on the Sobolev exponents). So,

$$\|(d^*_{(\exp i\xi)A} d_{(\exp i\xi)A} - \mathrm{d}^*_A \mathrm{d}_A)\xi_1\|_{L^2} \leq c\|a\|_{H^1}\|\xi_1\|_{H^2} \leq c\|\xi_1\|_{H^2}\|\xi\|_{H^2},$$

where the last inequality is by lemma 5.4. The constant c then depends on $\|a_0\|_{H^1(\overline{\Sigma})}$, where $A = \mathrm{d} + a_0$ - $\|a_0\|_{H^1(\overline{\Sigma}_R)}$ can be bounded above if $R \geq R_1$ and so c is R-independent.

The next term $\xi \mapsto (\mathcal{L}_{(\exp i\xi)u} - \mathcal{L}_u)$ is a smooth map. There is an L^∞ bound on its derivative, since $u(\mathbb{C} \backslash B_{R_0})$ is compact. Hence

$$\|\mathcal{L}_{(\exp i\xi)u} - \mathcal{L}_u\|_{C^0} < c\|\xi\|_{C^0} < c\|\xi\|_{H^2}.$$

By the Sobolev multiplication theorem,

$$\|(\mathcal{L}_{(\exp i\xi)u} - \mathcal{L}_u)\xi_1\|_{L^2} \leq c\|\xi\|_{H^2}\|\xi_1\|_{H^2}.$$

So, $\|D\mathcal{F}^R_\xi - D\mathcal{F}^R_0\| \leq c\|\xi\|_{H^2}$. For a small enough $\delta > 0$,

$$\|\xi\|_{H^2} < \delta \implies \|D\mathcal{F}^R_\xi - D\mathcal{F}^R_0\| \leq \frac{1}{2C}.$$

Step 3: *Finishing the proof*

To finish the proof of proposition 5.3, we apply the implicit function theorem (proposition A.4) on $\mathcal{F}$, with the values of δ and C found in the previous step. We assume that R is larger than the R_1 in Step 1. We need to ensure that

$$\|\mathcal{F}(0)\|_{L^2} < \frac{\delta}{4C}.$$

This can be done by taking a large enough value of R, since $\|{*F_A} + \Phi(u)\|_{L^2} < \infty$.

To prove the symmetry result (9) suppose ξ is the solution produced by the implicit function theorem. Then it is a unique solution in $B_\delta(H^2(\overline{\Sigma}, \mathfrak{k})) \cap \operatorname{Im} Q_R$.

But, by the symmetry of (A, u), $\tau_\gamma \xi$ is also a solution in B_δ and it is in $\operatorname{Im} Q_R$. So, $\xi = \tau_\gamma \xi$. $\square$

The following lemma was used in the proof of proposition 5.3.

Lemma 5.4 (Norm bound for $\mathcal{G}_{\mathbb{C}}$ action on $\mathcal{A}$)**.** *Let $R > 0$, and let $A = \mathrm{d} + a$ be a connection on the bundle $(\mathbb{C} \backslash B_R) \times K$ for which $\|a\|_{H^1(\mathbb{C}\backslash B_R)} < \epsilon$. For any $\xi \in H^2(\mathbb{C}\backslash B_R, \mathfrak{k})$ satisfying $\|\xi\|_{H^2} < 1$, $(\exp i\xi)A \in H^1$ and there is a constant $C(\epsilon)$ so that*

$$\|(\exp i\xi)A - A\|_{H^1(\mathbb{C}\backslash B_R)} \leq C\|\xi\|_{H^2(\mathbb{C}\backslash B_R)}.$$

Proof. The infinitesimal action of $i\xi$ on a connection A is $*\mathrm{d}_A\xi$. Suppose $a_t : [0,1] \to H^1_A(\mathfrak{k})$ is the solution of the ODE

$$\frac{da_t}{dt} = *d_{A+a_t}\xi = *\mathrm{d}_A\xi + *[a_t \wedge \xi], \quad a_0 = a.$$

Then, $(\exp i\xi)A = A + a_1$.

$$\frac{d}{dt}\|a_t\|_{H^1} \leq \|\frac{da_t}{dt}\|_{H^1} \leq c(\|\xi\|_{H^2} + \|a_t\|_{H^1}\|\xi\|_{H^2}) \leq c(1 + \|a_t\|_{H^1}),$$

since $\|\xi\|_{H^2} < 1$. The constant c depends on the norm of $\mathrm{d}_A : H^2 \to H^1$. Now, for any t, $\|a_t\| < c$, which implies $\frac{d}{dt}\|a_t\|_{H^1} \leq c\|\xi\|_{H^2}$. This proves the result. $\square$

We identify the complement of the orbifold point with $\mathbb{C}$, and let B_R denote an open ball of radius R around 0. The next two technical lemmas show that we can assume that a pair (A, u) is in *standard form* near infinity.

Proposition 5.5. (Standard form near infinity for orbifold connections) *Suppose A is a connection on $P \to \mathbb{P}(1, n)$. There is a complex gauge transformation g on P, and a trivialization of P over $\mathbb{C}$ so that $gA|_{\mathbb{C}} = d + \lambda d\theta$ on $\mathbb{C}\backslash B_R$ for some $R > 0$, $\lambda \in \mathfrak{k}$ satisfying $e^{2\pi n\lambda} = \mathrm{Id}$. Conversely, if A is a connection on $\mathbb{C}$ satisfying the above condition, then it extends to a connection on a principal bundle over $\mathbb{P}(1, n)$.*

Proof. Pick an $R_1 > 0$ and consider $B_{R_1} \subseteq \tilde{U}_1$. By, for example, theorem 1 in Donaldson [12], there is a unique complex gauge transformation e^{is}, $s : B_{R_1} \to \mathfrak{k}$, $s|_{\partial B_{R_1}} = 0$, so that $e^{is}A$ is a flat connection. and the complex gauge transformation e^{is} is unique up to multiplication by elements in $\mathcal{K}(B_{R_1})$. Notice that $(\sigma_n^* e^{is})\mu$ also transforms A to a flat connection. So, there is a gauge transformation $k : B_{R_1} \to K$ so that $ke^{is} = (\sigma_n^* e^{is})\mu$. Writing

$$(\sigma_n^* e^{is})\mu = \mu e^{i \operatorname{Ad}_\mu^{-1}(s\circ\sigma_n)},$$

we see that $s \circ \sigma_n = \operatorname{Ad}_\mu s$. Let $\eta : \tilde{U}_1 \to [0, 1]$ be a radially symmetric cut-off function that is 1 on B_R and vanishes on $\tilde{U}_1 \backslash B_{2R_1}$. It is easy to see that $e^{i\eta s}$ defines a complex gauge transformation g on all of $\mathbb{P}(1, n)$.

Now, we switch to working on U_2. Let $R = (2R_1)^{-n}$. Choose a trivialization $P|_{U_2} \to U_2 \times G$ so that gA is in radial gauge outside B_R, and since gA is flat, $gA = d + a(\theta)\mathrm{d}\theta$, for some $a : S^1 \to \mathfrak{k}$.

Claim. Let A_1 be a connection on $\mathbb{C} \times K$ that is equal to $\mathrm{d} + a(\theta)\mathrm{d}\theta$ on $\mathbb{C}\backslash B_R$, where $a : S^1 \to \mathfrak{k}$. There is a $\lambda \in \mathfrak{k}$ such that A_1 is gauge equivalent to a connection that is equal to $d + \lambda \mathrm{d}\theta$ on $\mathbb{C}\backslash B_R$.

Proof. Let $k : [0, 2\pi] \to K$ be the solution of

$$\frac{k^{-1}dk}{\mathrm{d}\theta} = a(\theta), \qquad\qquad k(0) = \mathrm{Id}\,.$$

The path $\theta \mapsto k(\theta)$ can be homotoped to a geodesic $\theta \mapsto e^{\lambda\theta}$, $\lambda \in \mathfrak{k}$. Then, $e^{\lambda\theta}k^{-1}$ is a gauge transformation on $\mathbb{C}\backslash B_R$ that is homotopic to the identity and it transforms A_1 to $d + \lambda\mathrm{d}\theta$ on $\mathbb{C}\backslash B_R$. By using a cut-off function, $e^{\lambda\theta}k^{-1}$ can be extended to a gauge transformation on all of $\mathbb{C}$. □

By this claim, $g(A)$ is gauge-equivalent to a connection that is $d + \lambda\mathrm{d}\theta$ on $\mathbb{C}\backslash B_R$. Its holonomy about infinity is $e^{2\pi\lambda}$. Since $g(A)$ has trivial holonomy for loops close to 0 in $\tilde{U}_1$, we have $e^{2\pi n\lambda} = \mathrm{Id}$.

For the converse, we construct a bundle $P \to \mathbb{P}(1, n)$. Set $\mu = e^{-2\pi\lambda}$ and $\tau = e^{n\lambda\theta}$. We are given $A|_{U_2}$. $A|_{\tilde{U}_1}$ can be constructed using the transition function τ. $A|_{\tilde{U}_1}$ will be trivial on $B_{R^{-n}} \subseteq \tilde{U}_1$. □

Remark 5.6. (a) (Choice of n) Given a connection A on $\mathbb{C} \times K$ of the form mentioned in the above proposition, i.e. $A = d + \lambda\mathrm{d}\theta$ on $\mathbb{C}\backslash B_R$ with $e^{2\pi\lambda n} = \mathrm{Id}$. We can extend A to a connection on principal bundles not just over $\mathbb{P}(1, n)$, but also $\mathbb{P}(1, mn)$ for any positive integer m.

(b) (Choice of λ) Topologically, the set of principal K-bundles $P \to \mathbb{P}^1$ is $\pi_1(K)$. This can be seen as follows : the deformation retract of the transition map $\mathbb{C}^* \to K$ is a loop $S^1 \to K$, whose homotopy type determines the bundle. The loop $S^1 \to K$ can be deformed to a geodesic loop $\theta \mapsto e^{\lambda\theta}$, where $\lambda \in \mathfrak{k}$ satisfies $e^{2\pi\lambda} = \mathrm{Id}$. Given a bundle $P \to \mathbb{P}(1, n)$, the transition maps would now produce a geodesic path $\theta \in [0, 2\pi] \mapsto e^{\lambda\theta}$, $\lambda \in \mathfrak{k}$ satisfies $e^{2\pi n\lambda} = \mathrm{Id}$. The bundle P is not altered by deforming the path $\theta \mapsto e^{\lambda\theta}$ (keeping the endpoints Id and $e^{2\pi\lambda}$ fixed) or by applying Ad_k to the path for some $k \in K$. In the lemma above, this classifying path is $\theta \mapsto e^{\lambda\theta}$. In the claim, we observed that this is equivalent to the homotopically equivalent path $\theta \mapsto k(\theta)$. In our description of bundles over $\mathbb{P}(1, n)$, this path can be recovered from the transition functions μ, τ as $\lim_{r\to 0}\{\theta \in [0, \frac{2\pi}{n}] \mapsto \mu^{-1}(r, 0)\mu(r, \theta)\}$. The limit path may not exist, but this determines a homotopy class of paths from Id to $\tau(0)$ in K. (Recall $\mu^{-1}(r, 0)\mu(r, \frac{2\pi}{n}) = \tau(r, 0)^{-1}$.)

The next result follows easily from proposition 5.5.

Proposition 5.7. (Standard form near infinity for gauged holomorphic maps) *Let $P \to \mathbb{P}(1, n)$ be a principal K-bundle and let (A, u) be a gauged holomorphic map from P to X. There is a complex gauge transformation g on P and a trivialization of P over $\mathbb{C}$ so that $g(A, u)$ satisfies the following:*

(a) *There is a* $\lambda \in \mathfrak{k}$ *so that* $gA = d + \lambda \mathrm{d}\theta$ *on* $\mathbb{C}\backslash B_R$ *for some* $R > 0$. *The element* λ *satisfies* $e^{2\pi n\lambda} = \mathrm{Id}$.
(b) *For any* $\theta \in [0, 2\pi)$, $\lim_{r\to\infty} e^{-\lambda\theta} u(r,\theta) = x$ *and* $e^{2\pi\lambda} x = x$.

Conversely, any gauged holomorphic map from $\mathbb{C} \times K$ *to* X *that satisfies the above conditions extends to a map on* $\mathbb{P}(1,n)$ *for some principal bundle* $P \to \mathbb{P}(1,n)$. n *can be taken to be any positive integer so that* $e^{2\pi n\lambda} = \mathrm{Id}$.

Proof of proposition 5.1. By proposition 5.7, we can assume that for some $R > 0$, $\lambda \in \mathfrak{k}$, $A = \mathrm{d} + \lambda \mathrm{d}\theta$ on $\mathbb{C}\backslash B_R$, and $\lim_{r\to\infty} e^{-\lambda\theta} u(r,\theta) = x$, and $x \in X^{\mathrm{ss}}$. Call the covering map $\pi_\Sigma : \tilde{\Sigma} \to \Sigma$. We denote $\pi_\Sigma^*(A,u)$ also by (A,u). On $\tilde{\Sigma}$, $A = \mathrm{d} + n\lambda \mathrm{d}\theta$

Step 1: Let $k_1 : \tilde{\Sigma} \to K$ be the gauge transformation $k_1(r,\theta) = e^{-n\lambda\theta}$. Then $k_1 A = \mathrm{d}$ on $\tilde{\Sigma}$ and $k_1 u$ extends holomorphically over $\tilde{\Sigma} \cup \{\infty\}$ with $u(\infty) = x$. We also observe that $k_1 \circ \sigma_n = e^{-2\pi\lambda} k_1$.

Step 2: *There is a holomorphic* $g_1 : \tilde{\Sigma} \cup \{\infty\} \to G$ *so that* $|\Phi(g_1 k_1 u)|_{L^2(\tilde{\Sigma})} < \infty$. g_1 *satisfies the symmetry relation* $g_1(e^{2\pi i/n} \cdot) = \gamma^{-1} g_1(\cdot) e^{2\pi\lambda}$, *where* $\gamma \in K$ *satisfies* $\gamma^n = \mathrm{Id}$.

We remark that $g_1 k_1 u$ extends holomorphically over ∞. We'll make $|\Phi(g_1 k_1 u)|_{L^2(\tilde{\Sigma})}$ finite by choosing g_1 so that $\Phi(g_1 k_1 u(\infty)) = 0$ and

$$\mathrm{d}(g_1 k_1 u)(\infty) \subseteq T_x \Phi^{-1}(0) \cap J T_x \Phi^{-1}(0).$$

Indeed, on $\tilde{\Sigma}$, $\Phi(g_1 k_1 u)$ and its derivative vanish at ∞, so $|\Phi(g_1 k_1 u)|_{L^2(\tilde{\Sigma})}$ will be finite.

Pick $s_0 \in \mathfrak{k}$ such that $e^{is_0} x \in \Phi^{-1}(0)$, denote $y = e^{is_0} x$. Apply the complex gauge transformation $g_1' \equiv e^{is_0}$ so that $\Phi \circ (g_1' k_1 u)$ vanishes at ∞. Set $\gamma = e^{is_0} e^{2\pi\lambda} e^{-is_0}$.

Next, we aim to find g_1'' with $g_1''(\infty) = \mathrm{Id}$ that makes the derivative of $\Phi \circ (g_1'' g_1' k_1 u)$ vanish at ∞. We need $g_1'' : \tilde{\Sigma} \cup \infty \to G$ to satisfy $g_1'' \circ \sigma_n = \gamma^{-1} g_1'' \gamma$. To satisfy both these conditions, we require that $\mathrm{d}g_1''(\infty) : \mathbb{C} \to \mathfrak{g}$ be complex-linear and satisfy the following: (to simplify notation, denote $u_1 = g_1' k_1 u$)

$$d(g_1'' u_1)(\infty) \subseteq T_y \Phi^{-1}(0) \cap J T_y \Phi^{-1}(0), \quad \mathrm{d}g_1''(\infty)(d\sigma_n(v)) = \mathrm{Ad}_\gamma^{-1} \mathrm{d}g_1''(\infty)(v) \tag{11}$$

for all $v \in T_\infty(\tilde{\Sigma} \cup \{\infty\})$. The first condition translates to

$$\mathrm{d}g_1''(\infty)(v)_y + du(v) \subseteq T_y \Phi^{-1}(0) \cap J T_y \Phi^{-1}(0).$$

The element $\mathrm{d}g_1''(\infty)$ is defined uniquely by this condition because $\mathfrak{g}$ acts freely on $T_y X$ and $T_y X = \mathfrak{g} y \oplus (T_y \Phi^{-1}(0) \cap J T_y \Phi^{-1}(0))$. Furthermore $\mathrm{d}g_1''(\infty)$ is complex-linear and it satisfies the second condition in (11) because $u_1 \circ \sigma_n = \gamma^{-1} u$. To produce g_1'' given $\mathrm{d}g_1''$, we use the following claim.

Claim. Let $\Gamma \subset G$ be a finite group and $\dim G = m$. There is a linear Γ action on $\mathbb{C}^m$ and a holomorphic Γ-equivariant chart $\overline{\phi} : U \to V \subseteq \mathbb{C}^m$ for a neighbourhood $U \subseteq G$ containing Id. Also $\overline{\phi}(\mathrm{Id}) = 0$. Γ acts on G by conjugation.

Proof of claim. Pick a holomorphic chart $\phi : U \to \mathbb{C}^m$ with $\phi(\mathrm{Id}) = 0$. Equip $\mathbb{C}^m$ with a Γ-action obtained by pushing forward the Γ-action via $d\phi(\mathrm{Id}) : \mathfrak{g} \to \mathbb{C}^m$. Define, for $y \in U$,

$$\overline{\phi}(y) := \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\gamma^{-1}\phi(\gamma y).$$

$\overline{\phi}$ is Γ-equivariant, and since $d\phi(0) = d\overline{\phi}(0)$, it is an isomorphism in a neighbourhood of 0. □

We use the above claim taking Γ to be the cyclic group generated by γ. Given $\mathrm{d}g_1''(\infty)$ and we have a Γ equivariant holomorphic chart $\overline{\phi} : U \to \mathbb{C}^m$ for a neighbourhood of $\mathrm{Id} \in G$. Define $\overline{\phi} \circ g_1''$ linearly by

$$\overline{\phi} \circ g_1''(z) := d(\overline{\phi} \circ g_1'')(\infty)\frac{1}{z}.$$

For a large enough R, g_1'' is defined on $\tilde{\Sigma}_R \cup \{\infty\}$. It is holomorphic and satisfies $g_1'' \circ \sigma_n = \gamma^{-1}g_1''\gamma$. Take $g_1 := g_1''g_1'$.

Step 3: *Finishing the proof*

Since g_1 found in Step 2 is holomorphic, it preserves the trivial connection k_1A (see remark 2.5). So far, we have on $\tilde{\Sigma}$, $g_1k_1(A,u) = (d, g_1k_1u)$. g_1k_1u extends holomorphically over ∞ and has the symmetry property $g_1k_1u(e^{2\pi i/n}\cdot) = \gamma g_1k_1u(\cdot)$, where $\gamma \in K$ satisfies $\gamma^n = \mathrm{Id}$.

We want to apply proposition 5.3 on $\overline{\Sigma}$. But when we pass to $\overline{\Sigma}$, we will no longer have $\Phi(u)$ in L^2, because the metric on $\overline{\Sigma}$ is 'bigger' than that of $\tilde{\Sigma}$. Our strategy is to apply a complex gauge transformation $e^{i\xi}$ that makes $\Phi(u)$ vanish near infinity. We will show that $\xi \in H^2(\tilde{\Sigma})$, so that the resulting connection is in $H^1(\tilde{\Sigma})$ and its curvature is in $L^2(\tilde{\Sigma})$. These conditions will be satisfied after passing to $\overline{\Sigma}$ (see (8)) so that proposition 5.3 can be applied. The details are as follows. The moment map Φ is real analytic so by step 2,

$$|\Phi \circ u| = \frac{a_2}{r^2} + \frac{a_3}{r^3} + \dots$$

close to ∞ on $\tilde{\Sigma}$. Since G acts locally freely on the semistable locus, there is a unique $\xi : \tilde{\Sigma} \to \mathfrak{k}$ so that $\Phi(e^{i\xi}u) \equiv 0$. The quotient $X/\!\!/G$ is compact, so $|\xi|$ has similar asymptotic behavior as $|\Phi(u)|$ close to ∞, so $\xi \in H^2(\tilde{\Sigma})$. By the symmetry of u, we can say that ξ satisfies $\xi(e^{2\pi i/n}\cdot) = \mathrm{Ad}_\gamma^{-1}\,\xi(\cdot)$. By lemma 5.4, $e^{i\xi}g_1k_1(A) = \mathrm{d} + a$ with $a \in H^1(\tilde{\Sigma})$. By (8), if on $\overline{\Sigma}$, $e^{i\xi}g_1k_1(A) = \mathrm{d} + a$, then $\|a\|_{H^1(\overline{\Sigma})} < \infty$ and $\|F_{\mathrm{d}+a}\|_{L^2(\overline{\Sigma})} < \infty$. Note the symmetry relation

$$e^{i\xi}g_1k_1(A,u)(e^{2\pi i}\cdot) = \gamma^{-1}e^{i\xi}g_1k_1(A,u)(\cdot).$$

Now, $e^{i\xi}g_1k_1(A,u)$ satisfies all the hypothesis of proposition 5.3, so there is a complex gauge transformation $g_2 \in H^2(\overline{\Sigma})$ that sends $e^{i\xi}g_1k_1(A,u)$ to a finite energy symplectic vortex on $\overline{\Sigma}$ and satisfies $g_2(e^{2\pi i}\cdot) = \gamma^{-1}g_2(\cdot)\gamma$.

Next, we construct a gauge transformation on $\mathbb{C}$. We have the symmetry

$$g_2 e^{i\xi} g_1 k_1(e^{2\pi i}\cdot) = \gamma^{-1} g_2 e^{i\xi} g_1 k_1(\cdot).$$

Suppose $\nu \in \mathfrak{k}$ is such that $e^{2\pi\nu} = \gamma$, then $re^{i\theta} \mapsto e^{\nu\theta}$ is a gauge transformation on $\overline{\Sigma}$. $e^{\nu\theta} g_2 e^{i\xi} g_1 k_1$ is invariant under 2π rotations and so it descends to a complex gauge transformation g on $\Sigma = \mathbb{C}\backslash B_R$. By remark 6.3, after modifying g by a gauge transformation, $g(A,u)$ is smooth on $\mathbb{C}\backslash B_R$. (Upstairs, this would amount to altering g_2 in a symmetric way.) Since both u and gu map to X^{ss} on $\mathbb{C}\backslash B_R$ and G acts locally freely on X^{ss}, g is also smooth on $\mathbb{C}\backslash B_R$. To produce a smooth complex gauge transformation on all of $\mathbb{C}$, we write $g = ke^{i\zeta}$, where $k \in \mathcal{K}(\Sigma)$ and $\zeta : \Sigma \to \mathfrak{k}$ are smooth. Let η be a cut-off function on $\mathbb{C}\backslash B_R$ that is 1 in $\mathbb{C}\backslash B_{2R}$ and 0 in the neighbourhood of ∂B_R. $e^{i\eta\zeta} \in \mathcal{G}(\mathbb{C})$ makes (A,u) a finite energy vortex on $\mathbb{C}\backslash B_{2R}$.

Finally, we look at the bundle P as a whole. Originally, P was described by transition functions $\mu = e^{-2\pi\lambda}$ and $\tau = e^{n\lambda\theta}$. We applied the transformation $g_2 e^{i\xi} g_1$ on $\tilde{U}_1$ and the descent of $e^{n\nu\theta} g_2 e^{i\xi} g_1 k_1$ on U_2. As a result of these transformations, P is now described by the transition functions $\mu = e^{-2\pi\nu}$ and $\tau = e^{n\nu\theta}$. The complex gauge transformation applied on $P \to \mathbb{P}(1,n)$ is smooth on $\mathbb{C}$. To look at its behaviour at ∞, we work on a neighbourhood in $\tilde{U}_1$, where the complex gauge transformation is given by $g_2 e^{i\xi} g_1$. g_1 and g_2 are smooth on $\tilde{U}_1$. Since $\xi \in H^2(\tilde{\Sigma})$, $e^{i\xi} \to \mathrm{Id}$ as $z \to \infty$. So, $e^{i\eta\zeta}$ is a smooth on $\mathbb{C}$ and extends continuously to a complex gauge transformation on $P \to \mathbb{P}(1,n)$. □

6. From a holomorphic map to a vortex : on the affine line

In this section we prove the first part of the main theorem 2.10. Given a gauged holomorphic map (A,u) on $\mathbb{C}$ that extends to a map over $\mathbb{P}(1,n)$, by proposition 5.1, there is a complex gauge transformation g on $\mathbb{C}$ so that $g(A,u)$ is a vortex on $\mathbb{C}\backslash B_{R_1}$. Next, we need to find complex gauge transform $g(A,u)$ so that it is a finite energy vortex on $\mathbb{C}$. For this, we use theorem 4.1, which for Riemann surfaces Σ with boundary, provides a complex gauge transformation $e^{i\xi}$ to change a gauged holomorphic map to a vortex. However, ξ is in H^2_δ, but the normal derivatives of ξ need not vanish on the boundary. This means, if we apply proposition 5.1 followed by theorem 4.1, the resulting connection A may not lie in $W^{1,p}_{\mathrm{loc}}$. So, the required result is obtained by applying theorem 4.1 to a sequence of compact sets that exhaust $\mathbb{C}$.

To prove theorem 2.10 (i), we need the following lemma on convergence of vortices:

Lemma 6.1 ([24], [35])**.** *Let $p > 2$ and Ω_i be a sequence of precompact sets exhausting $\mathbb{C}$:*

$$\Omega_1 \Subset \Omega_2 \Subset \cdots \Subset \mathbb{C} \qquad\qquad \bigcup_i \Omega_i = \mathbb{C}.$$

Suppose for each i, $(A_i,u_i) \in H^2 \times H^3_{\mathrm{loc}}$ is a vortex on Ω_i and

$$\sup_i E(\Omega_i,(A_i,u_i)) < \infty.$$

Then, passing to a subsequence, there are gauge transformations $k_i \in H^3(\Omega_i, K)$, a finite set $Z \subset \mathbb{C}$ and a finite energy vortex $(A_\infty, u_\infty) \in H^2 \times H^3_{\mathrm{loc}}$ over $\mathbb{C}$ such that

(a) $k_i A_i \rightharpoonup A_\infty$ *in* H^2 *on compact subsets of* $\mathbb{C}$*, and strongly in* $W^{1,p}$.
(b) $u_i \rightharpoonup u_\infty$ *in* H^3 *on compact subsets of* $\mathbb{C} \backslash Z$ *and strongly in* $W^{2,p}$.

Proof. This lemma is a combination of results in [34] and [24]. But we explain how one can get the result for higher regularity spaces than that in [34]. By the bounded energy condition, we have, for some constant c,

$$\|\mathrm{d}_{A_i} u_i\|_{L^2} < c.$$

Since the image of all the u_i is contained in a compact set ([34]), we get $\|\mathrm{d}_{A_i}\Phi(u_i)\|_{L^2}$ is bounded. Hence, by the vortex condition $*F_{A_i} + \Phi(u_i) = 0$, $\|\mathrm{d}_{A_i} * F_{A_i}\|_{L^2}$ is also bounded, that is,

$$\sup_{k \geq i} \|F_{A_k}\|_{H^1(B_i)} < \infty, \quad \forall i > 0.$$

By Uhlenbeck compactness for non-compact manifolds (theorem A' in [31]), there is a sequence of gauge transformations k_i in H^3 so that $k_i A_i \rightharpoonup A_\infty$ in H^2 on compact subsets of $\mathbb{C}$.

The rest of the proof is same as [24] - that there is a finite bubbling set Z, and away from this set, a subsequence of u_is converge in C^0 on compact subsets, and hence by elliptic regularity they converge weakly in H^3. The elliptic regularity in this context is standard (theorem 3.1 in [7]), but we present a short self-contained proof in lemma 6.2. The strong convergence statements follow from the compact inclusion $H^k(\Omega) \hookrightarrow W^{2,k-1}(\Omega)$, where $\Omega \subseteq \mathbb{C}$ is bounded. □

Lemma 6.2 (Elliptic regularity for gauged holomorphic maps)**.** *Let $s \geq 1$, $\Omega \subseteq \mathbb{C}$ be pre-compact and (A_i, u_i) be a sequence of gauged holomorphic maps such that $A_i \rightharpoonup A_\infty$ in $H^s(\overline{\Omega})$ and $u_i \to u_\infty$ in $C^0(\overline{\Omega})$, then $u_i \rightharpoonup u_\infty$ in $H^{s+1}(\overline{\Omega}')$, for any Ω', whose closure is contained in int(Ω).*

Proof. The proof is by induction on s. The base case is proved in the same way as the induction step. First note that it suffices to work locally in X: Pick an atlas $X = \cup_\alpha \mathcal{V}_\alpha$ such that $\mathcal{V}_\alpha$ is bi-holomorphic to an open subset of $\mathbb{C}^n$. Since $u_i \to u_\infty$ in C^0, we can find a finite cover $\Omega = \cup_\beta \mathcal{U}_\beta$ such that for $u_i(\mathcal{U}_\beta)$ is contained in a single $\mathcal{V}_\beta$ for large i. So, now we can think of u_i as mapping to $\mathbb{C}^n$.

In each chart we apply a combination of Sobolev multiplication and regularity theorems. As in [7], write $A_i = \mathrm{d} + \Phi_i dx + \Psi_i dy$, where Ψ_i, $\Phi_i \in H^s(\Omega, \mathfrak{k})$ and the holomorphicity equation for (A_i, u_i) is

$$-\overline{\partial} u_i = \Phi_i(u_i) + J_X \Psi_i(u_i) \tag{12}$$

We know that both A_i and u_i weakly converge in H^s, so Ψ_i, Φ_i and u_i are uniformly bounded in H^s. We next show that $\Phi_i(u_i)$ and $\Psi_i(u_i)$ are uniformly bounded in H^s. For this, recall $L \in \Gamma(\Omega, \mathrm{End}(\mathfrak{k}, T\Omega))$ defined by (7). $\Phi_i(u_i)$ can be seen as a

product $\Phi_i(u_i) = L(u_i)\Phi_i$. Since L is smooth, $\|L(u_i)\|_{H^s(\mathcal{U}_\alpha)} < c$ for all i, α. By Sobolev multiplication (proposition A.5), for $s \geq 1$,

$$\|L(u_i)\Phi_i\|_{H^s(\mathcal{U}_\alpha)} \leq c\|L(u_i)\|_{H^s(\mathcal{U}_\alpha)}\|\Phi_i\|_{H^s(\mathcal{U}_\alpha)}.$$

By holomorphicity, $\|\overline{\partial} u_i\|_{H^s(\mathcal{U}_\alpha)} < c$ for all i, α. By elliptic regularity for curves in $\mathbb{C}^n$,

$$\|u_i\|_{H^{s+1}(\mathcal{U}'_\alpha)} \leq c(\|\overline{\partial} u_i\|_{H^s(\mathcal{U}_\alpha)} + \|u_i\|_{L^2(\mathcal{U}_\alpha)}).$$

where $\overline{\mathcal{U}}'_\alpha \subseteq \mathcal{U}_\alpha$ and c depends on $\mathcal{U}_\alpha$, $\mathcal{U}'_\alpha$. By picking $\mathcal{U}'_\alpha$ such that they cover Ω', we obtain a uniform bound on $\|u_i\|_{H^{s+1}(\Omega')}$ and so $u_i \rightharpoonup u_\infty$ in $H^{s+1}(\Omega')$. □

Remark 6.3 (Regularity for vortices). Proposition D.2 in [34] shows that given a vortex $(A, u) \in W^{1,p}(\mathbb{C})$, $p > 2$, there is a gauge transformation $k \in W^{2,p}(\mathbb{C}, K)$ so that $k(A, u)$ is smooth. Using the above lemma this can be strengthened to produce $k \in H^2$ for $(A, u) \in H^1 \times C^0$. The proof in [34] also applies for vortices on $\mathbb{C}\backslash B_R$ - it applies to any non-compact Riemann surface which can be exhausted by compact sets that are deformation retracts.

The following 2 results are also needed for the proof of theorem 2.10 (i). The first one is proposition 4.3.1 in [29].

Proposition 6.4. *Let $p > 2$. Suppose Σ is a compact Riemann surface with boundary. Let $(A_i, u_i) \in \mathcal{H}(P, X)$ be a sequence such that $A_i \to A_\infty$ in $W^{1,p}$ and there is a finite set $Z \subseteq \Sigma$ so that $u_i \to u_\infty$ in C^1 on compact subsets of $\Sigma\backslash(Z \cup \partial\Sigma)$. Also, $F_i := *F(A_i) + u_i^*\Phi \to 0$ in L^p. Then, there exist constants C and i_0 so that for $i > i_0$, there is a complex gauge transformation $\exp i\xi_i$, $\xi_i \in W^{2,p}_\delta(\Sigma, P(\mathfrak{g}))$ so that $(\exp i\xi_i)(A_i, u_i)$ is a vortex and satisfies $\|\xi_i\|_{W^{2,p}} < C\|F_i\|_{L^p}$.*

The next one is proposition 4.3.2 in [29]. Roughly it says that in a complex gauge orbit, there is at most one vortex up to gauge. The proof is reproduced, because it will be useful in understanding the corresponding result for affine vortices.

Proposition 6.5. *Let (A_0, u_0), $(A_1, u_1) \in \mathcal{H}(P, X)$ be vortices on a compact Riemann surface Σ that are related by a complex gauge transformation g, i.e. $(A_1, u_1) = g(A_0, u_0)$ and assume $g(\partial\Sigma) \subseteq K$. Then, (A_0, u_0) and (A_1, u_1) are gauge-equivalent, i.e. $g \in \mathcal{K}$.*

Proof. After a gauge transformation, we can assume $(A_1, u_1) = e^{i\xi}(A_0, u_0)$, where $\xi \in \Gamma(\Sigma, P(\mathfrak{k}))$ and $\xi|_{\partial\Sigma} = 0$. Let $(A_t, u_t) := e^{it\xi}(A_0, u_0)$. We know $F_{A_0,u_0} = F_{A_1,u_1} = 0$. For $\xi|_{\partial\Sigma} = 0$,

$$\begin{aligned}\frac{d}{dt}\int_\Sigma \langle *F_{A_t,u_t}, \xi\rangle &= \int_\Sigma \langle \mathrm{d}^*_{A_t}\mathrm{d}_{A_t}\xi + u_t^* d\Phi(J(\xi)_{u_t}), \xi\rangle_{\mathfrak{k}} \\ &= \|\mathrm{d}_{A_t}\xi\|^2_{L^2} + \int_\Sigma \omega_{u_t}((\xi)_{u_t}, J(\xi)_{u_t}) \geq 0.\end{aligned}$$

The inequality is strict for non-zero ξ. So, $\xi = 0$ and (A_0, u_0) and (A_1, u_1) are gauge-equivalent. □

Proof of theorem 2.10 (i). Given a gauged holomorphic map (A, u) on $P \to \mathbb{P}(1, n)$, proposition 5.1 produces a complex gauge transformation e^{is} on P (that is smooth on $\mathbb{C}$ and continuous at ∞). $(A_0, u_0) := e^{is}(A, u)|_{\mathbb{C}}$ is a finite energy vortex on $\mathbb{C}\backslash B_R$. Next applying theorem 4.1, for any $i > R$, there is a complex gauge transformation $g_i \in W^{2,p}_\delta(B_i, G)$ such that $(A_i, u_i) := g_i(A_0, u_0)$ is a vortex on B_i. By modifying the g_is by gauge transformations, if necessary, we may assume that (A_i, u_i) is smooth (by theorem 3.1 in [7]). Also, recall that $E(A_i, u_i, B_i) \leq E(A_0, u_0, B_i)$.

Since $\{(A_i, u_i)\}$ are a sequence of vortices on B_i, which exhaust $\mathbb{C}$ and they satisfy an energy bound, they converge modulo bubbling to a finite energy vortex (A_∞, u_∞) on $\mathbb{C}$. i.e. By lemma 6.1, there is a subsequence of (A_i, u_i) (still denoted by the same subscripts), a sequence of gauge transformations $k_i \in H^3(B_i)$ and a finite set $Z \subseteq \mathbb{C}$ so that $k_i(A_i, u_i)$ converges to (A_∞, u_∞) in $W^{1,p} \times W^{2,p}(K)$ for all compact sets $K \subseteq \mathbb{C}\backslash Z$. For ease of notation we absorb the gauge transformations k_i into g_i and assume that (A_i, u_i) converges to (A_∞, u_∞) away from the bubbling set Z. Since (A_∞, u_∞) has finite energy, $\Phi(u_\infty(z)) \to 0$ as $z \to \infty$. So, we may assume, by increasing R if necessary, that $u_\infty(\mathbb{C}\backslash B_R) \subseteq X^{\mathrm{ss}}$. Also, we can assume $Z \subseteq B_R$ and so for large i, $u_i(\mathbb{C}\backslash B_R) \subseteq X^{\mathrm{ss}}$. In the next 3 paragraphs, we show that there is no bubbling. For this we focus our attention on $\overline{B}_{2R}$.

Since there is no bubbling on $\overline{B}_{2R}\backslash B_R$ and since $u_i(\overline{B}_{2R}\backslash B_R) \subseteq X^{\mathrm{ss}}$ for all i including $i = \infty$, there is a sequence of complex gauge transformations $g'_i \in W^{2,p}(\overline{B}_{2R}\backslash B_R)$ such that $g'_i(A_i, u_i) = (A_\infty, u_\infty)$ on $\overline{B}_{2R}\backslash B_R$. We remark that such g'_i exist because the action of G on X^{ss} is locally free, but the choice of g'_i may not be unique .

Write $g'_i = k'_i e^{i\xi'_i}$, where $k'_i : \overline{B}_{2R}\backslash B_R \to K$ and $\xi'_i : \overline{B}_{2R} \to \mathfrak{k}$. We know $u_i \to u_\infty$ in $W^{2,p}(\overline{B}_{2R}\backslash B_R)$. By applying lemma 6.6 and corollary 6.7, there is a subsequence of g'_i that converges to Id in $W^{2,p}(\overline{B}_{2R}\backslash B_R)$, and since (2) is a smooth function, $\xi'_i \to 0$ in $W^{2,p}(\overline{B}_{2R}\backslash B_R)$. Consider a cut-off function $\eta : \overline{B}_{2R}\backslash B_R \to [0, 1]$ that is 1 in the neighbourhood of ∂B_{2R} and 0 in the neighbourhood of ∂B_R. Then, $k'_i e^{i\eta\xi'_i}(A_i, u_i)$ is a gauged holomorphic map on $\overline{B}_{2R}$ that agrees with (A_∞, u_∞) close to the boundary. It satisfies $F_{k'_i e^{i\eta\xi'_i}(A_i,u_i)} \to 0$ in $L^p(B_{2R})$, because $F_{k'_i e^{i\eta\xi'_i}(A_i,u_i)}$ is non-zero only in $B_{2R}\backslash B_R$. So, we also have $F_{e^{i\eta\xi'_i}(A_i,u_i)} \to 0$ in $L^p(B_{2R})$. By proposition 6.4, for large i, there exist $\xi''_i \in W^{2,p}(B_{2R}, \mathfrak{k})$, such that $e^{i\xi''_i} e^{i\eta\xi'_i}(A_i, u_i)$ is a vortex. Proposition 6.4 also shows that $\xi''_i \to 0$ in $W^{2,p}(B_{2R})$.

We are now ready to show that $Z = \emptyset$. Since $\xi'_i, \xi''_i \to 0$ in $W^{2,p}(\overline{B}_{2R})$ and $A_i \to A_\infty$ in $W^{1,p}(\overline{B}_{2R})$, we get $e^{i\xi''_i} e^{i\eta\xi'_i} A_i \to A_\infty$ in $W^{1,p}(\overline{B}_{2R})$. By proposition 6.5, for all i, $e^{i\xi''_i} e^{i\xi'_i}(A_i, u_i)$ are in the same gauge orbit as each other. By standard elliptic regularity arguments (see [29, Lemma 4.3.4]) A_∞ is also in the same gauge orbit and if $k''_i e^{i\xi''_i} e^{i\eta\xi'_i} A_i = A_\infty$ for $k''_i \in W^{2,p}(\overline{B}_{2R})$, then $k_i \rightharpoonup \mathrm{Id}$ in $W^{2,p}(\overline{B}_{2R})$. $k_i \to \mathrm{Id}$ strongly in $C^1(\overline{B}_{2R})$. So, $g_i \to g_\infty$ in $C^1(\overline{B}_{2R})$, therefore $u_i = g_i u_0 \to g_\infty u_0$ in $C^1(\overline{B}_{2R})$. Since $g_\infty u_0 = u_\infty$, we have $Z = \emptyset$.

Since the action of G on X^{ss} is locally free, for $z \in \mathbb{C}\backslash B_R$ we may define $g_{>R}(z)$ to be an element such that $g_\infty(z)u_0(z) = u_\infty(z)$, unique up to right multiplication

by an element of the finite stabilizer $G_{u_0(z)}$. Given a choice of such $g_{>R}(z')$ for some $z' \in \mathbb{C}\backslash B_R$, this choice uniquely extends to $g_{>R}(z)$ for all $z \in C\backslash B_R$. Since $g_\infty(z)u_0(z)u_\infty(z)$ for $z \in B_{2R}$, this implies that g_∞ extends uniquely to a complex gauge transformation on all of $\mathbb{C}$ satisfying $g_\infty u_0 = u_\infty$. In addition $g_\infty A_0 = A_\infty$. The reason for this is as follows: for $x \in X^{\mathrm{ss}}$ close to $\Phi^{-1}(0)$, we have

$$\mathfrak{k}_x \cap J\mathfrak{k}_x = \phi. \tag{13}$$

Since $u_\infty(\infty) \in \Phi^{-1}(0)$, by increasing R, if necessary, we can assume that this condition is satisfied for all $u_\infty(c)$ for $c \in \mathbb{C}\backslash B_R$. Let $g_\infty A_0 - A_\infty = a$. Since both (gA_0, u_∞) and (A_∞, u_∞) are holomorphic, we have $a^{0,1}_{u_\infty} = 0$. Write $a = a_x dx + a_y dy$, a_x, $a_y : \mathbb{C}\backslash B_R \to \mathfrak{k}$. Then, $a_x(u) + Ja_y(u) = 0$ and by (13), $a = 0$ on $\mathbb{C}\backslash B_R$.

We look at the regularity of g_∞. Using proposition D.2 in [34], we can modify g_∞ by a gauge transformation in $W^{2,p}_{\mathrm{loc}}$, so that $g_\infty(A_0, u_0)$ is smooth. By applying lemma 6.8 in neighbourhoods in $\mathbb{C}$, we conclude that $g_\infty : \mathbb{C} \to G$ is smooth. Write $g_\infty = k_\infty e^{i\xi_\infty}$. We assumed (A_0, u_0) to be a finite energy vortex on $\mathbb{C}\backslash B_R$, which means $\lim_{r\to\infty} u_0(r,\theta) \in \Phi^{-1}(0)$ for any θ. The same is true for u_∞ also. This implies that $\xi_\infty(r,\theta) \to 0$ as $r \to \infty$.

We now have a complex gauge transformation $e^{i\xi_\infty}e^{is}$ on P that is smooth on $\mathbb{C}$ and continuous at ∞, we show that $e^{i\xi_\infty}e^{is} \in \mathcal{G}(P)_{\mathrm{we}}$. Given the finite energy vortex $e^{i\xi_\infty}e^{is}(A,u)$, theorem 2.10(ii) gives $g \in \mathcal{G}(P')_{\mathrm{we}}$ so that $ge^{i\xi_\infty}(A,u)$ is a smooth gauged holomorphic map on $P' \to \mathbb{P}(1,n)$. (The proof of part (ii) of the theorem is independent of part (i)). By using the arguments in the uniquess part of the proof of theorem 2.10(ii), P' and P are isomorphic, so we assume $P' = P$. The proof of uniqueness in (ii) still applies if we weaken the assumption g_1, $g_2 \in \mathcal{G}(P)_{\mathrm{we}}$, and instead just assume that g_1, g_2 are smooth on $\mathbb{C}$ and extend continuously over ∞. So, $ge^{i\xi_\infty}e^{is} \in \mathcal{G}(P)$ is smooth. Since $g \in \mathcal{G}(P)_{\mathrm{we}}$, the same is true for $e^{i\xi_\infty}e^{is}$ also. That the vortex $e^{i\xi_\infty}e^{is}(A,u)$ is unique up to gauge transformations, is proved separately in proposition 7.4 using theorem 2.10 (ii). □

Lemma 6.6, corollary 6.7 and lemma 6.8 were used in the proof of theorem 2.10 (i).

Lemma 6.6. *Suppose $\Omega \subseteq \mathbb{C}$ is compact and that the maps $u_i : \Omega \to X^{\mathrm{ss}}$ converge in $W^{2,p}$ to $u_\infty : \Omega \to X^{\mathrm{ss}}$. Also, we are given $g_i : \Omega \to G$ so that $g_i u_i = u_\infty$. Then, a subsequence of g_i converges in $W^{2,p}$ to g_∞.*

Proof. Let $x \in X/\!\!/G$ and $y \in \pi_G^{-1}(x)$. We can find a slice at y, i.e. a locally closed G_y-invariant submanifold $V \subset X^{\mathrm{ss}}$ containing y and so that there is an isomorphism $V \times_{G_y} G \to GV$. So, $V \times G \to GV$ is a $|G_y|$-cover. Suppose $U \subset \Omega$ is such that $u_\infty(U) \subset V$, $u_i(U) \subset V$ for large i. Ω can be covered by a finite number of sets of the form of U. Fix a lift $\tilde{u}_\infty = (v_\infty, G_\infty) : U \to V \times G$. Choose the lifts $\tilde{u}_i = (v_i, G_i)$ so that $G_\infty^{-1} G_i = g_i$. Since $u_i \to u_\infty$, a subsequence of $\tilde{u}_i$ converges in $W^{2,p}$. (This is because: $G \times V \to GV$ is a local diffeomorphism and has inverses locally. $\cup_i u_i(U) \subset V$ is pre-compact, and in this set the derivatives of the inverse of $G \times V \to GV$ are bounded.) So, G_i converges and consequently $g_i \to g_\infty$. Working

with the chosen cover of Ω, by successively passing to subsequences, we obtain a limit g_∞ on all of Ω. $\square$

Corollary 6.7. *In the above lemma, g_∞ stabilizes u_∞. So by replacing g_i by $g_\infty^{-1} g_i$, we can obtain a sequence of complex gauge transformations that converge to* Id *in $W^{2,p}$.*

Lemma 6.8. (Regularity of complex gauge transformations) *Let Ω be a compact Riemann surface and $P \to \Omega$ be a principal K-bundle, and $P_{\mathbb{C}} := P \times_K G$ a G-bundle. Suppose $A \in \mathcal{A}(P)$ is a smooth connection and $g \in \mathcal{G}(P)_{2,p}$ a complex transformation such that gA is also smooth. Then g is smooth.*

Proof. We use the relation on $P_{\mathbb{C}}$: $\overline{\partial}_{gA} = g \circ \overline{\partial}_A \circ g^{-1}$. $a := gA - A$ is smooth and

$$\overline{\partial}_{gA} - \overline{\partial}_A = a^{0,1} = g\overline{\partial}_A(g^{-1}) = -\overline{\partial}_A g g^{-1}.$$

Therefore $a^{0,1} g = -\overline{\partial}_A g$. The smoothness of g follows by elliptic bootstrapping. $\square$

7. From vortices to holomorphic maps

In this section, we prove theorem 2.10 (ii). The main tool used is the following result on the asymptotic behavior of a finite energy vortex (A, u) on $\mathbb{C}$.

Proposition 7.1. (Exponential Decay for Vortices) *Suppose G acts locally freely on X^{ss}. Let n be a positive integer such that for any $x \in X^{\mathrm{ss}}$, the order of the stabilizer group $|G_x|$ divides n. Let (A, u) be a finite energy vortex on $\mathbb{C}$ with target X. Then, for every $\epsilon > 0$, there is a constant C such that*

$$|F_A(z)|^2 + |\mathrm{d}_A u(z)|^2 + |\Phi(u(z))|^2 \leq C|z|^{-2-\frac{2}{n}+\epsilon}, \quad \forall z \in \mathbb{C} \backslash B_1.$$

The norms are taken with respect to the standard Euclidean metric on $\mathbb{C}$.

Ziltener [36] proves this result for $n = 1$, in appendix B, we explain how it generalizes to the case when $X /\!\!/ G$ is an orbifold. The following are conclusions of proposition 7.1. (proofs in appendix B).

Proposition 7.2. (Removal of singularity for vortices at infinity) *Assume the setting of proposition 7.1. Then, there exist $x_0 \in \Phi^{-1}(0)$ and $k_0 \in C^1([0, 2\pi], K)$ such that*

$$\lim_{r \to \infty} \max_{\theta \in [0,2\pi]} d(x_0, k_0(\theta) u(re^{i\theta})) = 0.$$

Suppose, the restriction of A in radial gauge to the circle $\{|z| = r\} \simeq S^1$ is $\mathrm{d} + a\mathrm{d}\theta$, for any $0 < \gamma < \frac{1}{n}$, there is a c such that for $r \geq 1$,

$$|k_0^{-1} \partial_\theta k_0 + a(r, \cdot)| < cr^{-\gamma}.$$

Necessarily, x_0 is fixed by $k_0(2\pi)$, which is of finite order.

Proof of theorem 2.10 (ii). Recall the notation $\Sigma_R = \mathbb{C} \backslash B_R$ and $\tilde{\Sigma}_R = \mathbb{C} \backslash B_{R^{1/n}}$ is its n-cover with covering map $z \mapsto z^n$. The value of R will be fixed in the course of the proof.

First, we show that, after a gauge transformation, $(\tilde{A}, \tilde{u}) := (A, u)(z^{-n})$, defined over $\mathbb{C}\backslash\{0\}$, extends in a weak sense to all of $\mathbb{C}$. Note that we are working on the chart $\tilde{U}_1 \simeq \mathbb{C}$ from 2.6. To $(\tilde{A}, \tilde{u})$, apply the gauge transformation $\tilde{k}(r, \theta) := k_0(n\theta)$ where k_0 is given by Proposition 7.2. Recall that k_0 was only defined on $[0, 2\pi]$, which can be extended as $k_0(\theta + 2\pi) = k_0(2\pi)k_0(\theta)$. This way, $\tilde{k}$ is a well-defined gauge transformation on $\mathbb{C}\backslash\{0\}$. By proposition 7.1, $\tilde{k}\tilde{u}$ extends continuously over ∞, with $\tilde{k}\tilde{u}(\infty) = x_0$. This makes $\tilde{u}$ lie in $W^{1,p}_{\text{loc}}(\tilde{U}_1)$. We claim that $\tilde{k}\tilde{A}$ also extends to an L^p connection over $\tilde{\Sigma}_R \cup \{\infty\}$. For this, we assume $\tilde{A}$ is in radial gauge, so $\tilde{A} = \mathrm{d} + a\mathrm{d}\theta$. Then,

$$\|\tilde{k}\tilde{A}\|^p_{L^p(B_1)} \approx \int_0^1 \int_0^{2\pi} |\frac{1}{r}(k_0^{-1}\partial_\theta k_0 + a(r^{-n}, \theta))|^p r\mathrm{d}\theta dr \leq c\int_0^1 r^{n\gamma p + 1 - p} dr,$$

which is finite by choosing $\gamma > \frac{1}{n}(1 - \frac{2}{p})$.

Next, we show that for some small $\tilde{R}$, $\tilde{k}\tilde{A}$ can be complex gauge-transformed to the trivial connection on $B_{\tilde{R}} \subset \tilde{U}_1$. For $0 < \tilde{R} < 1$, let $\sigma_{\tilde{R}} : B_1 \to B_{\tilde{R}}$ denote the dilation function $x \mapsto \tilde{R}x$. Letting $\tilde{k}\tilde{A} = \mathrm{d} + \tilde{a}$, we have $\sigma^*_{\tilde{R}}(\tilde{k}\tilde{A}) = \mathrm{d} + R\tilde{a}(R\cdot)$. Now

$$\|\sigma^*_R(\tilde{k}\tilde{A})\|_{L^p(B_1)} = \|\tilde{R}\tilde{a}(\tilde{R}\cdot)\|_{L^p(B_1)} = \tilde{R}^{1-\frac{2}{p}}\|\tilde{a}\|_{L^p(B_{\tilde{R}})}.$$

For some $\tilde{R} \in (0, 1]$, $\|\sigma^*_{\tilde{R}}\tilde{A}\|_{L^p(B_1)}$ is small enough that lemma 7.3 is applicable, which means that there is $\tilde{\xi} \in W^{1,p}_0(B_{\tilde{R}})$ and $\tilde{k}_1 \in W^{1,p}(B_{\tilde{R}}, K)$ so that $F_{e^{i\tilde{\xi}}\tilde{k}\tilde{A}} = 0$ and $\tilde{k}_1 e^{i\tilde{\xi}}\tilde{k}\tilde{A} = d$. Pick $\tilde{k}_1$ so that $\tilde{k}_1(0) = \mathrm{Id}$. We observe that $\sigma^*_n(\tilde{k}A) = k_0(2\pi)A$. By symmetry arguments similar to the proof of proposition 5.5, we see that $\tilde{\xi} \circ \sigma_n = \mathrm{Ad}_{k_0(2\pi)}\tilde{\xi}$ and $\tilde{k}_1 \circ \sigma_n = k_0(2\pi)\tilde{k}_1 k_0(2\pi)^{-1}$. Putting everything together, we have

$$\sigma^*_n(\tilde{k}_1 e^{i\tilde{\xi}}\tilde{k}) = k_0(2\pi)\tilde{k}_1 e^{i\tilde{\xi}}\tilde{k}.$$

We find a complex gauge transformation g over $\mathbb{C}$ (U_2 in the notation of 2.6) corresponding to $\tilde{k}_1 e^{i\tilde{\xi}}\tilde{k}$ on $B_{\tilde{R}}\backslash\{0\}$ and show that $g(A, u)$ extends to a smooth gauged holomorphic map on a bundle $P \to \mathbb{P}(1, n)$. Let $R = \tilde{R}^{-1/n}$. Suppose $k_0 : [0, 2\pi] \to K$ is homotopic to the geodesic $\theta \mapsto e^{-\lambda\theta}$ with $k_0(2\pi) = e^{-2\pi\lambda}$. $e^{n\lambda\theta}\tilde{k}_1 e^{i\tilde{\xi}}\tilde{k}$ descends to a complex gauge transformation g on $\mathbb{C}\backslash B_R$ - it satisfies $e^{n\lambda\theta}\tilde{k}_1 e^{i\tilde{\xi}}\tilde{k}(z) = g(\frac{1}{z^n})$. The map $g : \mathbb{C}\backslash B_R \to G$ is homotopic to identity. This is because: on $B_{\tilde{R}}\backslash\{0\} \subset \tilde{U}_1$, both $\tilde{k}^{-1}\tilde{k}_1 e^{i\tilde{\xi}}\tilde{k}$ and $e^{n\lambda\theta}\tilde{k}$ are homotopic to identity. Let $g_t : \mathbb{C}\backslash B_R \to G$ be a deformation of g to Id, i.e. $g_0 \equiv \mathrm{Id}$ and $g_1 = g$. Let η be a cut-off function on $\mathbb{C}\backslash B_R$ that vanishes in the neighbourhood of ∂B_R and is 1 in the neighbourhood of ∂B_{2R}, then $g_\eta(A, u)$ satisfies the conditions in proposition 5.7 - i.e. on $\mathbb{C}\backslash B_{2R}$, $g_\eta A = \mathrm{d} + \lambda\mathrm{d}\theta$ and $\lim_{r\to\infty} e^{-\lambda\theta} g_\eta u(r, \theta) = x_0$. So, $g_\eta(A, u)$ extends to a gauged holomorphic curve over $\mathbb{P}(1, n)$. $g : \mathbb{C}\backslash B_R \to G$ is smooth because both u and gu are smooth and $u(\mathbb{C}\backslash B_R) \subset X^{\mathrm{ss}}$, where G acts locally freely. So, g_η is smooth on $\mathbb{C}$. It follows from the construction of $\tilde{k}_1 e^{i\tilde{\xi}}\tilde{k}$ that, $g_\eta(A, u)$ extends to a smooth gauged holomorphic map over $P \to \mathbb{P}(1, n)$ described by transition functions $\mu = e^{-2\pi\lambda}$ and $\tau = e^{n\lambda\theta}$.

This indeed proves the statement of the theorem. g_η can be written as $g_\eta = e^{i\zeta}k$, where $k \in \mathcal{K}(\mathbb{C})$ and $\zeta : \mathbb{C} \to \mathfrak{k}$ are smooth. We claim that $e^{-i\zeta} \in \mathcal{G}(P)_{\mathrm{we}}$, so that we can say that $k(A,u)$ extends to a weak gauged holomorphic map over $P \to \mathbb{P}(1,n)$. Let $\tilde{\zeta} : \tilde{U}_1 \to \mathfrak{k}$ be defined as $\tilde{\zeta}(z) = \zeta(\frac{1}{z^n})$. The complex gauge transformation $e^{-i\zeta}$ on U_2 corresponds to $e^{-n\lambda\theta}e^{-i\zeta}e^{n\lambda\theta}$ on $B_{\tilde{R}/2^n} \subset \tilde{U}_1$. By straightforward calulations, this is equal to $\tilde{k}_1 e^{-i\xi}\tilde{k}_1^{-1}$, which is in $W^{1,p}(B_{\tilde{R}/2^n})$. Since $k(A,u)$ extends to a weak gauged holomorphic map over P, the same can be said about (A,u), since k just contributes to the choice of trivialization of $P|_{\mathbb{C}}$.

Uniqueness: Given a finite energy vortex (A,u), the bundle $P \to \mathbb{P}(1,n)$ is determined uniquely by the path $\{\theta \in [0, \frac{2\pi}{n}] \mapsto k_0(\theta)\}$ up to homotopy and the action of Ad_k for $k \in K$ (see remark 5.6). k_0 is determined using proposition 7.1, the choice of this equivalence class is unique because it has to satisfy the condition

$$\lim_{r\to\infty} \max_{\theta\in[0,2\pi]} d(x_0, k_0(\theta)u(re^{i\theta})) = 0, \quad \text{where } x_0 = \lim_{r\to\infty} u(r,0) \in \Phi^{-1}(0).$$

Now suppose $g_1, g_2 \in \mathcal{G}(P)_{\mathrm{we}}$ so that $g_i(A,u)$ is a smooth gauged holomorphic map on P. On $B_{\tilde{R}} \subset \tilde{U}_1$, $g_1 u$, $g_2 u$ are smooth maps to X^{ss}, where G acts locally freely. So, $g_1^{-1}g_2$ is smooth in this region and hence $g_1^{-1}g_2 \in \mathcal{G}(P)$. □

The following lemma is used in the proof of theorem 2.10 (ii).

Lemma 7.3. *Let $A = \mathrm{d} + a$ be a connection on the trivial K-bundle over $B_1 \subseteq \mathbb{C}$ so that $a \in L^p(B_1)$. There is a constant c_1 such that if $\|a\|_{L^p(B_1)} < c_1$, there is $\xi \in W_0^{1,p}(B_1, \mathfrak{k})$ satisfying $F_{e^{i\xi}A} = 0$.*

Further, there is a gauge transformation $k \in W^{1,p}(B_1, K)$ so that $ke^{i\xi}A = \mathrm{d}$. k is unique up to left multiplication by a constant element in K.

Proof. The proof is by implicit function theorem on the function

$$\mathcal{F} : W_0^{1,p}(B_1,\mathfrak{k}) \to W^{-1,p}(B_1,\mathfrak{k}), \quad \xi \mapsto *F_{e^{i\xi}A}.$$

The linearization at ξ

$$D\mathcal{F}(\xi) = d^*_{e^{i\xi}A} d_{e^{i\xi}A} : W_0^{1,p}(B_1,\mathfrak{k}) \to W^{-1,p}(B_1,\mathfrak{k}). \tag{14}$$

We recall that $W^{-1,p} = (W_0^{1,p^*})^*$, where W_0^{1,p^*} is the subspace of W^{1,p^*} sections whose boundary trace vanishes, $\frac{1}{p} + \frac{1}{p^*} = 1$ and the dual is taken under the L^2 pairing. The operator $\mathrm{d}^*\mathrm{d} : W_0^{1,p}(B_1,\mathfrak{k}) \to W^{-1,p}(B_1,\mathfrak{k})$ is an isomorphism (see theorem 4.7 in [31]). If $A = \mathrm{d} + a$ and $a \in L^p$, then

$$(\mathrm{d}_A^*\mathrm{d}_A - \mathrm{d}^*\mathrm{d})\xi = *[a \wedge *d\xi] + d^*[a \wedge \xi] + *[a \wedge *[a \wedge \xi]]. \tag{15}$$

By the multiplication theorem (proposition A.5), this is compact. In $W^{1,p}$, $\ker \mathrm{d}_A^*\mathrm{d}_A$ consists of horizontal sections with respect to A, so $W_0^{1,p} \cap \ker \mathrm{d}_A^*\mathrm{d}_A = \{0\}$, so (14) is an isomorphism. By (15), $\|\mathrm{d}_A^*\mathrm{d}_A - \mathrm{d}^*\mathrm{d}\| \le c\|a\|_{L^p}$. If $\|a\|_{L^p}$ is small enough,

$$\|(\mathrm{d}_A^*\mathrm{d}_A)^{-1}\| \le 2\|(\mathrm{d}^*\mathrm{d})^{-1}\|.$$

In the statement of the implicit function theorem (proposition A.4), we take $C := 2\|(\mathrm{d}^*\mathrm{d})^{-1}\|$.

Similar to Step 2 in the proof of proposition 5.3, we can obtain a constant $\delta > 0$ such that for $\|\xi\|_{1,p} < \delta$,

$$\|D\mathcal{F}_\xi - D\mathcal{F}_0\| \leq \frac{1}{2C}.$$

By proposition A.4, if $\|F_A\|_{-1,p} < \frac{\delta}{4C}$, we get the result. This can be ensured if $\|a\|_{L^p}$ is small enough because by the multiplication theorem,

$$\|F_A\|_{-1,p} \leq c(\|a\|_{L^p} + \|a\|_{L^p}^2).$$

We next prove the second statement of the lemma. First note that a flat connection is gauge-equivalent to a connection that is smooth away from the boundary ∂B_1. Indeed, this is proved in [30] (theorem 3.1) for a base manifold without boundary, but since we only want interior regularity, the proof is identical. We briefly outline this proof: For a smooth connection A, if we pick a smooth connection A_0 close enough to A (in the L^p norm), there is a gauge transformation $k \in W^{1,p}$ that puts A in coulomb gauge with respect to A_0, that is,

$$\mathrm{d}^*_{A_0}(kA - A_0) = 0.$$

This is the content of the local slice theorem (theorem F') in [31]. Now, one has control over da and d^*a, using elliptic bootstrapping as in [30], it can be shown that kA is smooth away from ∂B_R. Similar to the first statement in the lemma, here we rely on the ellipticity of $\Delta : W_0^{1,p} \to W^{-1,p}$.

Finally note that a smooth flat connection on a contractible set is gauge equivalent to the trivial connection. The uniqueness statement in the lemma is obvious because the trivial connection is preserved only by constant gauge transformations. □

The following is the last component in the proof of the main theorem 2.10. To state it, we assume the result of part (ii) of this theorem.

Proposition 7.4. (At most one vortex in a complex gauge orbit) *Suppose* (A_0, u_0) *and* (A_1, u_1) *are finite energy vortices on* $\mathbb{C}$ *that extend to a weak gauged holomorphic map over a bundle* $P \to \mathbb{P}(1, n)$. *Suppose they are related by a complex gauge transformation* $g \in \mathcal{G}(P)_{\mathrm{we}}$. *Then* g *is a unitary gauge transformation, i.e.* $g \in \mathcal{K}(P)_{\mathrm{we}}$.

Proof. g can be written as $g = ke^{i\xi}$, where $k \in \mathcal{K}(P)_{\mathrm{we}}$ and $\xi \in \Gamma(\mathbb{P}(1,n), P(\mathfrak{k}))_{\mathrm{we}}$. We can assume $k = \mathrm{Id}$ and $(A_1, u_1) = e^{i\xi}(A_0, u_0)$. The proof proceeds in the same way as the corresponding result for vortices on a compact Riemann surface (proposition 6.5). Let $(A_t, u_t) := e^{it\xi}(A, u)$ for $t \in [0, 1]$. We write

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{C}} \langle *F_{A_t,u_t}, \xi\rangle &= \int_{\mathbb{C}} \langle \mathrm{d}^*_{A_t}\mathrm{d}_{A_t}\xi + u_t^* d\Phi(J(\xi)_{u_t}), \xi\rangle_{\mathfrak{k}} \\ &= \|\mathrm{d}_{A_t}\xi\|_{L^2}^2 + \int_{\mathbb{C}} \omega_{u_t}((\xi)_{u_t}, J(\xi)_{u_t}) + \lim_{r\to\infty}\int_{\partial B_r} \langle \mathrm{d}_{A_t}\xi, \xi\rangle_{\mathfrak{k}}.\end{aligned} \tag{16}$$

However, for the above computation to make sense, we need to show that the terms $\|\mathrm{d}_{A_t}\xi\|_{L^2(\mathbb{C})}$ and $\int_{\mathbb{C}} \omega_{u_t}((\xi)_{u_t}, J(\xi)_{u_t})$ are finite and the boundary term vanishes.

STEP 1: $\|d_{A_t}\xi\|_{L^2} < \infty$ *for* $t \in [0,1]$.
We work on the chart containing infinity - this is $\tilde{U}_1$ from 2.6. Consider $B_{\tilde{R}} \subset \tilde{U}_1$, for some $\tilde{R} > 0$. We have $A_0 \in L^p(B_{\tilde{R}})$ and $\xi \in W^{1,p}(B_{\tilde{R}})$. Repeating the calculations in lemma 5.4 for different Sobolev spaces, we can say $A_t - A_0 \in L^p(B_{\tilde{R}})$. Since $p > 2$, $\|d_{A_t}\xi\|_{L^2(B_{\tilde{R}})}$ is finite. Let $R = (\tilde{R})^{-1/n}$. The norms $\|d_{A_t}\xi\|_{L^2(\mathbb{C}\backslash B_R)}$ and $\|d_{A_t}\xi\|_{L^2(B_{\tilde{R}})}$ are equal, so both are finite, and hence $\|d_{A_t}\xi\|_{L^2} < \infty$ for $0 \le t \le 1$.

STEP 2: $\int_{\mathbb{C}} \omega_{u_t}((\xi)_{u_t}, J(\xi)_{u_t}) < \infty$ *for* $t \in [0,1]$.
We use asymptotic decay of vortices (proposition 7.1) to obtain an asymptotic bound on $|\xi|$: fix an $0 < \epsilon < \frac{2}{n}$, let $\delta = \frac{2}{n} - \epsilon$, then there is a constant c so that for $z \in \mathbb{C}\backslash B_1$,

$$e_{A,u}(z) \le c|z|^{-2-\delta}.$$

Since $X/\!\!/G$ is compact, there are constants c, ϵ so that for any $x \in \{|\Phi| < \epsilon\}$,

$$c^{-1}\langle s_1, s_2\rangle_{\mathfrak{k}} \le \langle (s_1)_x, (s_2)_x\rangle_{T_xX} \le c\langle s_1, s_2\rangle_{\mathfrak{k}} \tag{17}$$

for s_1, $s_2 \in \mathfrak{k}$. As a consequence, there is a constant c such that for any $x \in \Phi^{-1}(0)$ and $s \in \mathfrak{k}$ such that $e^{is}x \in \{|\Phi| < \epsilon\}$,

$$c^{-1}|s| < |\Phi(e^{is}x)| < c|s|.$$

We assume R is large enough that $u_0(\mathbb{C}\backslash B_R)$, $u_1(\mathbb{C}\backslash B_R) \subset \{|\Phi| < \epsilon\}$. Using asymptotic decay of $\|\Phi(u_0)\|_{L^2}$ and $\|\Phi(u_1)\|_{L^2}$, we conclude

$$|\xi(z)|^2 < c|z|^{-2-\delta}. \tag{18}$$

Since $u_t(\mathbb{C}\backslash B_R) \subset \{|\Phi| < \epsilon\}$ for $t = 0, 1$, by (17) and some straightforward calculations, we can conclude the same for all $t \in [0,1]$. The bound $|\xi(z)| < c|z|^{-2-\delta}$, together with (17), shows $|\omega_{u_t}((\xi)_{u_t}, J(\xi)_{u_t})(z)| < c|z|^{-2-\delta}$. Hence

$$\int_{\mathbb{C}} \omega_{u_t}((\xi)_{u_t}, J(\xi)_{u_t}) < \infty, \quad 0 \le t \le 1.$$

STEP 3: $\lim_{r\to\infty} \int_{\partial B_r} \langle \mathrm{d}_{A_t}\xi, \xi\rangle_{\mathfrak{k}}$ *for* $t \in [0,1]$.
In step 1, we showed that A_t is in $L^2(\mathbb{C}\backslash B_R)$, but this isn't enough to say anything about $A_t|_{\partial B_r}$. We now obtain a C^0-bound under suitable local trivializations of P. For this we cover $\mathbb{C}\backslash B_R$ by identical open sets : let $S \subseteq \mathbb{C}$ be an open set with smooth boundary such that

$$[-\frac{3}{4}, \frac{3}{4}] \times [-\frac{3}{4}, \frac{3}{4}] \subseteq S \subseteq [-1,1] \times [-1,1].$$

Then, $\{S + (x,y) : |x|, |y| \ge R - 2\}$ is a cover of $\mathbb{C}\backslash B_R$. Let $S'' \Subset S' \Subset S$ be such that their translates also cover $\mathbb{C}\backslash B_R$. The quantity $\langle \mathrm{d}_{A_t}\xi, \xi\rangle_{\mathfrak{k}}$ is gauge-invariant, so on each $S + (x,y)$, we can choose a different trivialization to study it. We focus on a single set $S_{xy} := S + (x,y)$ and let $r := \sqrt{x^2 + y^2}$. In the following discussion,

the constant c is independent of (x,y) and r. Fix a trivialization of $P|_{S_{xy}}$ so that $A_0 = \mathrm{d} + a_0$ is in Uhlenbeck gauge i.e.

$$\|a_0\|_{W^{1,p}(S_{xy})} < c\|F(A_0)\|_{L^p(S_{xy})} < c. \tag{19}$$

Under this trivialization, we write $A_t = \mathrm{d} + a_t$. By applying a gauge transformation $k : S_{xy} \to K$, we can put A_1 in Uhlenbeck gauge - i.e. if $kA_1 = \mathrm{d} + \tilde{a}_1$ then,

$$\|\tilde{a}_1\|_{W^{1,p}(S_{xy})} < c\|F(A_1)\|_{L^p(S_{xy})} < c. \tag{20}$$

Denote $g = ke^{i\xi}$, so $gA_0 = \mathrm{d} + \tilde{a}_1$. As in the proof of lemma 6.8, we can write $a^{0,1}g = -\overline{\partial}_{A_0} g$, where $a = \tilde{a}_1 - a_0$. By elliptic regularity,

$$\begin{aligned}\|g\|_{W^{1,p}(S'_{xy})} &\leq c(\|\overline{\partial} g\|_{L^p(S_{xy})} + \|g\|_{L^p(S_{xy})})\\ &\leq c(\|a^{0,1}g\|_{L^p(S_{xy})} + \|a_0^{0,1}g\|_{L^p(S_{xy})}) + c\|g\|_{L^\infty(S_{xy})})\end{aligned}$$

There is a L^∞ bound on g from (18) and the fact that K is compact. Together with the bounds on a and a_0 (from (19) and (20)), this shows that $\|g\|_{W^{1,p}(S'_{xy})} \leq c$. By applying elliptic regularity again, we can show $\|g\|_{W^{2,p}(S''_{xy})} \leq c$. Since (2) is an isomorphism,

$$\|\xi\|_{W^{2,p}(S''_{xy})} \leq c. \tag{21}$$

Repeating the calculations in lemma 5.4 for different Sobolev spaces, we get

$$\|a_t - a_0\|_{W^{1,p}(S''_{xy})} < c.$$

Together with the bound on a_0, this uniformly bounds $\|a_t\|_{W^{1,p}(S_{xy})}$. By Sobolev embedding $\|a_t\|_{C^0(S_{xy})} < c$ for $0 \leq t \leq 1$.

Consider the integral $\int_{\partial B_r} \langle \mathrm{d}_{A_t}\xi, \xi\rangle_{\mathfrak{k}}$. Partition the curve ∂B_r at points $x_1, \dots, x_n$, so that the segments $x_i x_{(i+1) \bmod n}$ are in a single S''_{xy},

$$\int_{\partial B_r} \langle \mathrm{d}_{A_t}\xi, \xi\rangle_{\mathfrak{k}} = \sum_{i=1}^n \left(\int_{x_i}^{x_{i+1}} \langle d\xi, \xi\rangle_{\mathfrak{k}} + \int_{x_i}^{x_{i+1}} \langle [a_t, \xi], \xi\rangle_{\mathfrak{k}} \right)$$

The first term evaluates to $\sum_i \langle \xi(x_{i+1}), \xi(x_{i+1})\rangle_{\mathfrak{k}} - \langle \xi(x_i), \xi(x_i)\rangle_{\mathfrak{k}}$. By a change of trivialization $\langle \xi(x_i), \xi(x_i)\rangle_{\mathfrak{k}}$ is not altered because $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$ is Ad_K-invariant. So, the first terms for all i sum to zero. The second term goes to zero as $r \to \infty$ because of the C^0 bound on a_t and the asymptotic bound on ξ (18).

STEP 4: *Finishing the proof.*
We have shown that the equation (16) is meaningful and the boundary term vanishes as $r \to \infty$. So,

$$\frac{d}{dt} \int_{\mathbb{C}} \langle *F_{A_t,u_t}, \xi\rangle > 0$$

if $\xi \neq 0$. Since $F_{A_0,u_0} = F_{A_1,u_1} = 0$, this implies that $\xi = 0$. $\square$

Appendix A. Some analytic results

In this section, we collect some analytic results used in the proof. The first is a modification of a theorem of Calderón (see theorem V.3 p.135 in [26]) which says that $\mathrm{Id} + \Delta : W^{2,p}(\mathbb{C}, \mathbb{C}) \to L^p(\mathbb{C}, \mathbb{C})$ is an isomorphism for $p > 1$.

Proposition A.1. *Let $\Sigma = \mathbb{C} \backslash B_1$. The operator $\mathrm{Id} + \Delta : H^2_\delta(\Sigma) \to L^2(\Sigma)$ is an isomorphism.*

Remark A.2. $H^2_\delta = \{f \in H^2 : f|_{\partial\Sigma} = 0\}$. To keep track of signs, we recall $\Delta = -(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) = \mathrm{d}^*\mathrm{d}$

We give a proof of this variation so that it is easy to see how it applies to an n-cover of Σ as well. (see remark A.3)

Proof of proposition A.1. First, we show that $\mathrm{Id} + \Delta : H^1_0 \to H^{-1}$ is an isomorphism. Recall that $H^{-1} = (H^1_0)^*$ via the L^2-pairing. For any f, $g \in H^1_0$

$$\langle (\mathrm{Id} + \Delta) f, g\rangle_{L^2} = \langle f, g\rangle_{L^2} + \langle df, dg\rangle_{L^2} = \langle f, g\rangle_{H^1}$$

So, $(\mathrm{Id} + \Delta) : H^1_0 \to (H^1_0)^*$ is given by $f \mapsto \langle f, \cdot\rangle_{H^1_0}$. By the Riesz representation theorem, this is an isomorphism.

Let $f \in L^2(\Sigma) \hookrightarrow H^{-1}$. Then, there is a $g \in H^1_0$ so that $(\mathrm{Id} + \Delta)g = f$. By elliptic regularity on bounded sets, $g \in H^2_{\mathrm{loc}}(\Sigma)$. It remains to bound g in $H^2(\Sigma)$. For this, we apply elliptic regularity on a collection of bounded sets that cover Σ. Let $S \subseteq \mathbb{C}$ be an open set with smooth boundary such that $[-\frac{3}{4}, \frac{3}{4}] \times [-\frac{3}{4}, \frac{3}{4}] \subseteq S \subseteq [-1, 1] \times [-1, 1]$. Let $\Omega := B_3 \backslash B_1 \subseteq \mathbb{C}$. Then, $\{S + (x, y) : |x|, |y| \geq 2\} \cup \{\Omega\}$ is a cover of Σ. Pick $0 < \epsilon < 0.1$. Let $\Omega' := B_{3-\epsilon} \backslash B_1$. Let $S' \subset S$ be such that $\{S' + (x, y) : |x|, |y| \geq 2, x, y \in \mathbb{Z}\} \cup \{\Omega'\}$ is also a cover of Σ'. There is a constant $c_{S,S'}$ depending only on S, S' so that for any integers $|x|, |y| \geq 2$,

$$\|g\|_{H^2(S'+(x,y))} \leq c_{S,S'}(\|f\|_{L^2(S+(x,y))} + \|g\|_{H^1(S+(x,y))}), \tag{22}$$

Since $g|_{\partial B_1} = 0$,

$$\|g\|_{H^2(\Omega')} \leq c_{\Omega,\Omega'}(\|f\|_{L^2(\Omega)} + \|g\|_{H^1(\Omega)}), \tag{23}$$

Adding (23) and (22) for all x, $y \in \mathbb{Z}$, we get

$$\|g\|_{H^2(\Sigma)} \leq c(\|f\|_{L^2(\Sigma)} + \|g\|_{H^1(\Sigma)}) \leq c'\|f\|_{L^2(\Sigma)},$$

where c, c' are independent of f. □

Remark A.3. The above theorem still holds if $\Sigma = \mathbb{C} \backslash B_1$ is replaced by an isometric n-cover $\overline{\Sigma} = \{re^{i\theta} : r \geq 1, \theta \in \mathbb{R}/2n\pi\mathbb{Z}\}$. This is because the covering by $S + (x, y)$ can be lifted to $\overline{\Sigma}$. Then, all but a bounded part of $\overline{\Sigma}$, can be covered by identical-looking sets.

The other results presented here are very standard. The following is proposition A.3.4 in [20].

Proposition A.4. (Implicit function theorem) *Let $F : X \to Y$ be a differentiable map between Banach spaces. $DF(0)$ is surjective and has a right inverse Q, with $\|Q\| \leq c$. For all $x \in B_\delta$, $\|DF(x) - DF(0)\| < \frac{1}{2c}$. If $\|F(0)\| < \frac{\delta}{4c}$, then $F(x) = 0$ has a solution in B_δ. x is the unique solution in B_δ satisfying $x \in \operatorname{Im} Q$.*

Proposition A.5 (Sobolev multiplication)**.** *Let $\Omega \subseteq \mathbb{R}^n$, not necessarily compact. The multiplication operator*

$$W^{s_1,p_1}(\Omega) \times W^{s_2,p_2}(\Omega) \to W^{s_3,p_3}$$
$$(f,g) \mapsto fg$$

is bounded if $s_1 + s_2 \geq 0$, $s_3 \leq \min(s_1, s_2)$ and $s_3 - \frac{n}{p_3} < s_1 - \frac{n}{p_1} + s_2 - \frac{n}{p_2}$.

If Ω is compact, $s_1 + s_2 > 0$ and $s_3 < \max(s_1, s_2)$, the operator is compact.

Appendix B. Asymptotic decay for vortices for orbifold target

Proposition 7.1 is a consequence of a decay result for vortices on a cylinder (proposition B.2) and a result about the limit behaviour of u as $z \to \infty$ (proposition B.3).

Definition B.1 (Energy density)**.** Suppose (A, u) is a vortex from the principal bundle $P \to \Sigma$ to X. The *energy density* of (A, u) is

$$e_{(A,u)}(z) := \frac{1}{2}\left(|F_A(z)|^2 + |\mathrm{d}_A u(z)|^2 + |\Phi(u(z))|^2\right)$$

for any $z \in \Sigma$. The norms are defined in terms of the metric ω_Σ on Σ.

Proposition B.2 (Decay for vortices on the half cylinder, [36], theorem 1.3)**.** *Let Σ be a half cylinder*

$$\Sigma := \{(s,t) : s \geq 0, t \in \mathbb{R}/a\mathbb{Z}\}$$

for some $a > 0$ with an admissible metric $\omega_\Sigma = \lambda^2 ds \wedge dt$ where $\lambda = \lambda(s,t)$ is a positive function. Suppose G acts freely on X^{ss} and (A, u) is a finite energy vortex from the trivial bundle $\Sigma \times K$ to X such that $\overline{u(\Sigma)}$ is compact. Then, for every $\epsilon > 0$, there is a constant C such that

$$e_{(A,u)}(s + it) \leq C\lambda^{-2} e^{(-\frac{4\pi}{a} + \epsilon)s}.$$

Proposition B.3. *Suppose (A, u) is a finite energy vortex on the half cylinder Σ (described in proposition B.2). Let $\pi_G : X^{\mathrm{ss}} \to X /\!\!/ G$ denote the projection. Then, $\lim_{s\to\infty} \pi_G \circ u$ exists.*

Proof. The proof of proposition B.3 uses a combination of results from [36] and [20]. The main difficulty is that the action of G on X^{ss} is not free. [36] considers vortices where this action is free. $\pi_G \circ u$ is a J-holomorphic curve on $X /\!\!/ G$, but we can't use the results of [20] directly because $X /\!\!/ G$ has orbifold singularities.

For the mean value inequality, we use lemma 3.3 in [36]. The proof of this result works when the acton of K on $\Phi^{-1}(0)$ is locally free. In the setting of proposition B.2, this gives : there exists a number $s_0 \geq \frac{1}{2}$ such that for $z = (s,t) \in \{s \geq s_0\}$,

$$e_{(A,u)}(z) \leq \frac{32}{\pi} E((A,u), B_{\frac{1}{2}}(z)).$$

Since $E((A,u),\Sigma)$ is finite, the right side goes to 0 as $s \to \infty$. So, for large enough s_0, $\Phi(u(s,t))$ is close enough to 0 and so $u(s,t) \in X^{\mathrm{ss}}$, and $u_G := \pi_G \circ u$ is well-defined and is a holomorphic curve on $X/\!\!/G$. Also,

$$\begin{aligned}\ell(u_G(\{s=s_1, t \in \mathbb{R}/a\mathbb{Z}\})) &\leq \int_0^a |du_G(s_1,t)|dt \\ &\leq \int_0^a |\mathrm{d}_A u(s_1,t)|dt \to 0 \quad \text{ as } s_1 \to \infty\end{aligned} \tag{24}$$

Now, we switch to working on $X/\!\!/G$ to prove the result. For every $p \in X/\!\!/G$, there is a neighbourhood U_p and a uniformizing chart (V_p, G_p, π) such that $V_p \subset \mathbb{C}^m$, G_p is a finite group acting on U_p and $\pi : U_p \to V_p$ induces a homeomorphism from U_p/G_p to V_p (see [6]). Each U_p has a G_p-invariant symplectic form that descends to the symplectic form on $X/\!\!/G$. $X/\!\!/G$ is compact, so it can be covered by a finite number of such neighbourhoods $U_1, \ldots, U_k$. Fix a constant $\delta_0 > 0$ such that for any $p \in X/\!\!/G$, $B_p(\delta) \subset U_i$ for some $i \in \{1, \ldots, k\}$. If the length of the loop $\gamma : S^1 \to X/\!\!/G$ is less than δ_0, it can be lifted to the cover in some uniformizing chart and the isoperimetric inequality (Theorem 4.4.1, [20]) can be applied. The rest of the proof can be completed in the same way as the proof of the removal of singularities result for J-holomorphic curves in [20] (theorem 4.1.2). We need the second paragraph of the proof of lemma 4.5.1 (this requires Stokes' theorem for orbifolds, which can be proved by passing to a cover locally using regularizing charts), followed by the proof of theorem 4.1.2. Note that we we don't require holomorphic extension of u_G over the singularity. □

Proposition 7.1 now follows in a straightforward way.

Proof of proposition 7.1. Map $\mathbb{C}\backslash B_1$ to the half cylinder Σ setting $a = 2\pi$ and the change of coordinates $\Sigma \ni z \mapsto e^z \in \mathbb{C}\backslash B_1$. By proposition B.3, $u_G(\infty) := \lim_{s\to\infty} u_G(s,t)$ exists. Pick $x \in \pi_G^{-1}(u_G(\infty))$, and let S be a slice for the G-action at x. This means $G \times_{G_x} S \to X^{\mathrm{ss}}$ is a diffeomorphism onto its image. $\pi : G \times S \to X^{\mathrm{ss}}$ is a $|G_x|$-cover, equip $G \times S$ with $\pi^*\omega_X$ and $\pi^*\omega_X$. The left K-action is free and has moment map $\pi^*\Phi$. n divides G_x, so for some large s_0, $u(\Sigma_{s>s_0}) \subseteq GS$ and it lifts to $\overline{u} : \overline{\Sigma}_{s>s_0} \to G \times S$. Here $\overline{\Sigma}_{s>s_0} = \{(s,t) : s \geq s_0, t \in \mathbb{R}/2\pi n\mathbb{Z}\}$ is an n-cover of $\Sigma_{s>s_0}$. Now, proposition B.2 can be applied to the lift $(\overline{A}, \overline{u}) : \overline{\Sigma}_{s\geq s_0} \to G \times S$, and this proves proposition 7.1. □

Proof of proposition 7.2. We work in cylindrical co-ordinates. Map $\mathbb{C}\backslash B_1$ to the half cylinder $\Sigma = \{(s,t) : s \geq 0, t \in [0, 2\pi)\}$, with change of coordinates $\Sigma \ni z \mapsto e^z \in \mathbb{C}\backslash B_1$. The Euclidean metric on $\mathbb{C}\backslash B_1$ pulls back to $\omega_\Sigma = e^{2s} ds dt$ on Σ. The connection A can be put in radial gauge, so that on Σ, $A = \mathrm{d} + a(s,t)dt$. If $F_A = f ds \wedge dt$, then $\frac{\partial}{\partial s} a = f$. By proposition 7.1, for any $\epsilon > 0$, there is a C so that

$$|\frac{\partial}{\partial s} a| = |f| \leq C e^{(-\frac{1}{n} + \frac{\epsilon}{2})s}.$$

Fix an $\epsilon < \frac{2}{n}$. There exists $a_\infty \in C^0(\mathbb{R}/2\pi\mathbb{Z}, \mathfrak{k})$ so that

$$|a(s,t) - a_\infty(t)| \leq c_\epsilon e^{(-\frac{1}{n}+\frac{\epsilon}{2})s}$$

for all s, t. The bound on $\mathrm{d}_A u$ coming from proposition 7.1 can be written as

$$|\frac{\partial}{\partial s}u| + |\frac{\partial}{\partial t}u + a_u| \leq Ce^{(-2-\frac{1}{n}+\frac{\epsilon}{2})s}.$$

The bound on $|\frac{\partial}{\partial s}u|$ implies that $\lim_{s\to\infty} u(s,t) = u_\infty(t)$, where $u_\infty \in C^0(S^1, X)$. Since $\pi_G \circ u$ has a limit at ∞, u_∞ maps to a single G-orbit. The decay of $\Phi(u)$ implies that u_∞ maps to $\Phi^{-1}(0)$, and hence to a single K-orbit. Let $k_0 : [0, 2\pi] \to K$ be such that $k_0(t)u_\infty(0) = u_\infty(t)$. Observe that $k_0(2\pi)$ stabilizes $u_\infty(0)$. Finally, the bound on $|\frac{\partial}{\partial t}u + a_u|$ implies that $a_\infty = -k_0^{-1}\partial_\theta k_0$. This also means that $k_0 \in C^1(S^1, \mathfrak{k})$. $\square$

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School of Mathematics, Tata Institute of Fundamental Research, Colaba, Mumbai-400005, India.

E-mail address: sushmita@math.tifr.res.in

Mathematics-Hill Center, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ-08854, USA.

E-mail address: ctw@math.rutgers.edu